The Feedback Loop of Statistical Discrimination

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We study a dynamic model of interactions between a firm and job applicants to identify mechanisms that drive long-term discrimination. In each round, the firm decides which applicants to hire, where the firm's ability to evaluate applicants is imperfect. Each applicant belongs to a group, and central to our model is the idea that the firm becomes better at evaluating applicants from groups in which they have hired from in the recent past. We establish the firm's *initial evaluation ability* for each group to be a critical factor in determining long-term outcomes. We show that there is a threshold for which if the firm's initial evaluation ability for the group is below the threshold, the group's hiring rate decreases over time and eventually goes to zero. If the group starts above the threshold, then the hiring rate stabilizes to a positive constant. Therefore, even when two groups are identical in size and underlying skill distribution, a marginal difference in the firm's initial evaluation ability can lead to persistent disparities that exacerbate over time through a feedback loop. Importantly, the dynamic nature of our model allows us to assess the *long-term* impact of interventions, specifically, whether an improvement is sustained even after the intervention is lifted. In this light, we show that drastic short-term interventions are more effective compared to milder long-term interventions. Additionally, we show that smaller groups face inherent disadvantages, requiring a higher initial evaluation ability to achieve a favorable long-term hiring outcome and experiencing lower hiring rates even when they do.

1. Introduction

The application of machine learning algorithms in decision-making processes has expanded into many high-stakes areas, including hiring, lending, healthcare, and criminal justice (Barocas et al. 2023). Given the significant impact these decisions can have, important concerns have been raised regarding possible harms due to algorithmic biases. For example, a machine learning tool used by Amazon for recruiting was found to explicitly discriminate against women (Dastin 2022), and Obermeyer et al. (2019) showed that an algorithm used to allocate hospital resources were leading to inadvertent discrimination against African American patients.

These examples and others have drawn significant attention to the problem of algorithmic bias, which has inspired a growing literature on designing "fair" algorithms. Common algorithmic approaches include enforcing parity constraints across groups such as *demographic parity* (e.g., Zemel et al. (2013)), which mandates equal selection proportions across groups in decisions like hiring or college admissions, or ensuring that algorithms are "blind" to sensitive attributes such as race and gender (e.g., Dwork et al. (2012)). Another prominent approach focuses on ensuring that predictions are properly calibrated across groups (e.g., Hébert-Johnson et al. (2018)).

These approaches define "fairness" in the context of a *one-shot* setting (e.g., a prediction problem). However, in reality, algorithmic systems operate within *dynamic* contexts, and the dynamics can play a key role in perpetuating undesirable disparities. Consider the following examples.

1. Algorithmic hiring: A company's algorithmic hiring tool is trained on historical data from a workforce dominated by a specific group (e.g., a specific gender, graduates from a particular college, etc.). The algorithm is more effective at evaluating applicants from this group, leading to further inclination to hire from this majority group in the future.

2. Predictive policing: Predictive policing systems forecast crime in specific neighborhoods, leading to concentrated patrols in those areas. Crimes detected through heightened patrols are fed back into the system, further reinforcing the designation of these neighborhoods as high-risk.

3. Loan denials: A bank uses an algorithm to predict the default risk of loan applicants, which predicts that groups with lower socioeconomic status are at high-risk. Their loan applications are denied, and as a result, these groups lose opportunities to invest or create businesses, perpetuating their economic disadvantage.

These examples describe undesirable feedback loop mechanisms that can arise in several applications, and such issues cannot be formalized under static models. This suggests the need to study dynamic models that can capture delayed impact and feedback effects. This is the motivation for this paper, which focuses on the hiring application described in the first example.

Discrimination in hiring. Labor markets have consistently grappled with disparities that have persisted over time. In the United States, both gender and racial pay gaps have not meaningfully improved over the last couple of decades[1]. Our paper proposes a dynamic model that formalizes a feedback loop mechanism that explain the persistence of discrimination, and this model can then be used to provide insights on how to mitigate discrimination.

We build upon the literature on *statistical discrimination*, which aim to explain how differences in labor market outcomes between groups can arise even when firms act rationally in their hiring process. Specifically, we extend the canonical static model of Phelps (1972) by incorporating

[1] The hourly earnings for black men were 73% of white men in both 1980 and 2015 (https://www.pewresearch.org/short-reads/2016/07/01/racial-gender-wage-gaps-persist-in-u-s-despite-some-progress/). The gender pay gap was 80% in 2002 and 82% in 2022. (https://www.pewresearch.org/social-trends/2023/03/01/the-enduring-grip-of-the-gender-pay-gap/)

dynamics inspired by machine learning practices. Our model studies a firm making hiring decisions over time from applicants who are divided into groups. The key idea of our model is:

A firm is better at evaluating applicants from groups it has hired from in the recent past.

This concept is formalized through a noisy signal observed by the firm for each applicant: the signal is less noisy for groups in which the firm has recently hired from. This dynamic reflects the nature of machine learning systems, where more data enhances predictive accuracy, but older data becomes stale and less relevant.

Under our model, we evaluate the long-term hiring rate for each group. Our main result is that there are *multiple* distinct long-term outcomes that can arise, and which one emerges simply depends on the firm's *initial evaluation ability* for the group. That is, two identical groups may converge to completely different hiring rates if the firm initially had a slightly better ability to evaluate applicants from one group over the other. The resulting dynamics illustrate that historical disadvantages can persist and exacerbate over time through a feedback loop, leading to long-term disparities.

Our dynamic model also allows us to formalize the idea of interventions that are successful *in the long term*. Specifically, we say that an intervention that aims to help a disadvantaged group is successful in the long term if, compared to pre-intervention, the outcomes of the group improves *after the intervention is lifted*. For example, an intervention that enforces a temporary boost in the hiring rate of a group may improve outcomes in the short run, but this may not persist once the intervention is lifted. Our model and results suggest that a successful intervention in the long term should be *drastic*, even if it is in place for a short period of time.

This paper helps to bridge the literatures of statistical discrimination and algorithmic fairness. We extend classical models of statistical discrimination by introducing dynamics that shed light on how disparities evolve over time. On the algorithmic fairness front, our paper provides a *framework* that formalizes how algorithmic performance shifts with data availability influenced by past decisions. This framework can be used to evaluate long-term outcomes, which can be applied to settings beyond the hiring context.

Model summary. We introduce a dynamic model of a selective labor market composed of a firm and applicants, where applicants are clustered into "groups" (based on gender, race, college, etc.). For convenience, this paper uses terminology related to the labor market (a *firm* that *hires* applicants), but the model also readily applies to an admissions problem (e.g., a *college* that *admits* applicants). Each applicant has a true "skill", a scalar value, and the firm hires all applicants whose skill is expected to be above a fixed threshold. The firm's evaluation ability is imperfect and dependent on the observation of a noisy signal of each applicant's skill, where the variance of the noise is group-specific and dependent on past decisions. Specifically, we assume that the noise variance of the signal is a decreasing function of the number of hires from that group in the previous time step. This dynamic models the idea that a firm's ability to evaluate applicants from a group improves with recent hiring from that group, as past hires provide both performance feedback and contribute to future hiring decisions. This feedback loop, however, relies on recent hires, as older data and employee turnover diminish its impact over time. Under these dynamics, we evaluate how the hiring rate (percentage of applicants hired) for a group evolves over time. If the hiring rate eventually converges, we call this a steady state. We call an "active" steady state one where the corresponding hiring rate is positive, which implies that a constant fraction of applicants from that group are consistently hired. An "inactive" steady state is one where the hiring rate is zero.

1.1. Summary of Results

We now summarize our main results.

Convergence to two steady states. We show that there are exactly two possible steady state outcomes that a group can converge to, where one of them is active and the other is inactive. We characterize which of these steady states arise as a simple threshold on the initial noise variance. If the initial noise variance is below the threshold, the group converges to the active state, and otherwise, they converge to the inactive state. Hence, two groups that start with a small difference in their initial noise variances may converge to two completely different long-term hiring outcomes, even if their underlying skill distributions are identical.

Group size. Next, we show that groups of smaller sizes are fundamentally disadvantaged in two ways. First, the noise variance threshold needed to converge to the active steady state is smaller for the smaller group. Second, even if two groups are both in the active steady state, the *percentage* of applicants hired (which normalizes for group size) will be lower for the smaller group.

Interventions. For a group to converge to the active steady state, the noise variance must be reduced below a critical threshold. Once this threshold is crossed, the group benefits from a self-reinforcing feedback loop that sustains improved hiring outcomes even after the intervention ends. Thus, a sufficiently drastic intervention that substantially lowers the noise variance can lead to long-term improvements, even once the intervention is lifted. In contrast, milder interventions that do not achieve this threshold have only temporary effects and fail to produce lasting change. Notably, we show that enforcing a specific form of demographic parity (equalizing hiring rates across groups) is always sufficient to trigger convergence to the favorable steady state, if one exists. Specifically, the equalized hiring rate must match that of a group in the active state. If this intervention is unsuccessful, this serves as evidence that the favorable state does not exist, and hence *no* intervention that decreases the noise variance can be successful.

Competition across groups. We extend our model to one where two groups are competing for a limited number of positions at a single firm, and we characterize which steady states can arise under this model. We first show that if the initial hiring rate for one group is sufficiently small, the resulting steady state will be inactive for that group, and the firm will eventually exclusively hire from the other group. We show that such an exclusive steady state can arise even when initial hiring rates are equal, under uneven group sizes. When the total hiring capacity is small enough, then the smaller group will converge to the inactive steady state even if it started with the same hiring rate as the other group. On the positive side, we show that if the total capacity is large enough and the initial hiring rates have a small discrepancy, the system will converge to an active steady state for both groups.

We consider an extension where the underlying population size increases, which intensifies competition. We show that this exacerbates disparities: the group with higher evaluation ability secures a larger share of hires as the population grows.

Simulating a practical algorithmic hiring system. To complement our theoretical analysis, we conduct a simulation study in which a firm deploys a realistic machine learning (ML) based hiring system over time. In the simulation, each applicant has a feature vector, and their skill is a linear function of the features. The firm uses past hiring data to fit a predictor of their skill to make hiring decisions. The simulation results validate several key qualitative features of our model: the existence of two distinct steady states (active and inactive), the critical role of initial evaluation ability in determining long-run outcomes, and the disadvantage faced by smaller populations. Specifically, we show that the hiring rate converges either to zero or to a positive constant depending on the firm's initial estimate quality, mirroring the bifurcation behavior predicted in our theoretical model. These results highlight that the feedback loop dynamics we model are not merely a theoretical artifact, but persist under more realistic settings with noisy prediction and limited feedback.

Extensions. We consider several modeling extensions to check the robustness of our main results. We consider an extension where applicants can improve their skills by exerting effort. Exerting effort incurs a cost, and applicants exert the effort level that maximizes the chance they are hired minus the effort cost. We show that the results regarding the existence and convergence to two steady states continue to hold in this extension. Next, we also consider an extension of the model in which the noise variance can depend on more than one previous time step, as well as another variant where we impose a cap on the maximum value that the noise variance can take. Lastly, we consider an alternative noise update rule which is motivated by a model wherein which a firm observes multiple features for each applicant, and the firm aims to learn which feature is informative of the applicant's skill. We show that our main results hold under these extensions.

1.2. Implications

Persistence of historical discrimination. Our main result is that there are distinct steady states that are associated with drastically different hiring outcomes, and the outcome that arises depends critically on the firm's initial evaluation ability for each group. This mechanism helps explain real-world patterns, such as the persistent under-representation of certain demographic groups in leadership roles (Ospina and Foldy 2009, Krivkovich et al. 2024, Nagpaul 2024), or the continued dominance of graduates from specific elite institutions in competitive industries (Gerber and Cheung 2008, Witteveen and Attewell 2017). It highlights how historical disadvantages, reflected in initial evaluation disparities, can persist and even worsen over time, leading to entrenched long-term inequalities in hiring outcomes without any explicit bias in the firm's decision-making process.

Need for drastic interventions. Consider an intervention whose goal is to help a disadvantaged group. For example, an intervention may increase the number of applicants hired from the group, or a firm may spend more effort on evaluating the applicants to decrease the noise (e.g., by conducting more interviews). We say that an intervention is successful if the group converges to the active steady state. Our convergence result shows that a successful intervention must cause the noise variance of that group to fall below a critical threshold. This highlights the importance of *drastic* interventions. If an intervention is able to reduce a group's noise below the critical threshold, the group will *naturally* converge to the active state, maintaining its improved status even after the intervention ends. In contrast, mild interventions, even those in place for a long time, that do not lower the noise variance below the threshold are ineffective in the long term. Such interventions might temporarily boost hiring while they are in effect, but fail to create the self-reinforcing dynamics necessary for sustained improvement, causing the group to fall back to the inactive state when the intervention is lifted.

Our results can provide insight into the (in)effectiveness of real-world interventions. For example, the recent United States Supreme Court decision in *Students for Fair Admissions v. Harvard* in 2023 banned the use of affirmative action in college admissions, effectively removing an intervention that had been in place for decades. Preliminary post-ban data[2] indicate that Black enrollment has declined at three-quarters of colleges, with varying impacts across institutions. Applying our framework to this context, one possible interpretation is that affirmative action, while impactful during its implementation, may not have led to a state where an improvement can be naturally sustained. Of course, we acknowledge that this is a simplification of a highly complex issue, and our model captures one of many aspects of the broader systemic forces at play.

[2] https://edreformnow.org/2024/09/09/tracking-the-impact-of-the-sffa-decision-on-college-admissions/

Smaller groups are inherently disadvantaged. Our model reveals that smaller groups face inherent disadvantages in achieving favorable long-term hiring outcomes. Specifically, smaller groups require a higher initial evaluation ability to converge to the active steady state, making them more vulnerable to falling into the inactive (zero-hiring) state. Moreover, even when both large and small groups achieve active steady states, the hiring rate for the smaller group remains lower due to the weaker feedback loop generated by fewer hires. This stands in contrast to the common perception that parity in hiring rates across groups is the desirable or "fair" outcome—a notion often formalized through demographic parity constraints in the algorithmic fairness literature (e.g., Calders and Verwer (2010), Edwards and Storkey (2015), Zafar et al. (2017), Agarwal et al. (2018), Zemel et al. (2013)). Our results suggest that such parity may not naturally emerge even in the absence of bias, when group sizes differ.

1.3. Related Literature

As mentioned, our paper builds upon the literature on statistical discrimination. We describe the most relevant works from this literature, and we refer the reader to the excellent surveys of Fang and Moro (2011), Lang and Spitzer (2020), and Onuchic (2022) for a comprehensive literature review.

Multi-armed bandit models. Conceptually, our paper is closest related to recent works that study discrimination using multi-armed bandit models (Li et al. 2020, Bai et al. 2022, Komiyama and Noda 2024). These models capture how a firm's evaluations improve as a function of past hiring, which aligns with our central premise that hiring from a group enhances the firm's ability to assess future applicants from that group. A key distinction is our work is in treatment of past data: bandit models assume that all past data remains relevant indefinitely, and we assume that only recent data informs the firm's current evaluation ability. In our model, the firm must consistently hire from a group to maintain a low noise level when evaluating its applicants. This difference yields distinct insights. Specifically, our model differentiates between the timing and intensity of interventions: hiring 100 applicants in one round is not equivalent to hiring one applicant each over 100 rounds. In contrast, the cumulative nature of learning in bandit models makes these two interventions equivalent. Moreover, our framework highlights that *reducing the noise variance* is the key to achieving lasting improvement. This can be achieved not only through hiring more individuals from a group, but also through more accurate evaluations (e.g., by spending additional effort during interviews).

Relation to Phelps Model. Next, our paper is also closely related to papers that study the impact of heterogeneity in information quality as a mechanism for statistical discrimination, as in the seminal work of Phelps (1972), and extensions (e.g., Aigner and Cain (1977), Lundberg and Startz

(1983)). The model in our paper can be seen as a dynamic generalization of Phelps (1972); our work relies on the same *source* of discrimination (heterogeneous information quality), but we focus on studying how discrimination *evolves over time*. Other than being a dynamic model, a second departure of our work from the Phelps model is that we assume the firm makes *binary hiring decisions*, rather than *wage decisions*. This is a crucial and necessary feature of our model, since the idea that "firms get better in hiring from those they have hired from before" is only applicable when considering hiring decisions (and not when the firm hires everybody and only makes wage decisions). Studying binary decisions is the natural formulation of the problem from a machine learning perspective, since this corresponds to the ubiquitous classification problem.

Cornell and Welch (1996) studies a variation of Phelps (1972) that studies information heterogeneity across groups of applicants, but they consider hiring decisions rather than wage decisions (as in this paper). They assume that for each applicant, a firm receives a certain number of independent signals of quality, where the number of signals they receive is a function of their group. They assume the firm only hires *one* applicant, and they show that the hired applicant is more likely to come from the group in which the firm has more information about, and this probability approaches 1 when the number of applicants goes to infinity. Under multiple generations, they calculate the expected duration until the *first time* the hired applicant comes from the disadvantaged group. Although Cornell and Welch (1996) comments on the persistence of discrimination over time, they do not model the *dynamics* of how the underlying hiring mechanism evolves over time, which this paper focuses on.

More recently, Emelianov et al. (2022) studies a static model where the variance of the noise in the applicant's signal differ across groups. They study two variants in which the decision maker either knows or do not know the variance, and show that both can lead to discrimination. They also consider interventions similar to demographic parity and analyze its effects. Our paper can be seen as a dynamic generalization of this work.

Relation to Arrow Model. Our paper is also related to the seminal model of Arrow (1973), which introduces a model where groups are ex-ante identical, applicants make costly skill investment decisions, and firms provide higher wages to those with the investments.

The model is shown to have multiple equilibria, and hence, discrimination is explained as different groups arriving at distinct equilibria (called "coordination failure"). This formalizes the idea of "self-fulfilling stereotypes" — if a firm believes that a group will not make skill investments, the group, correspondingly, will not be incentivized to do so.

At a high level, our results have a similar flavor to that of the Arrow model. Specifically, we show that our model exhibits *multiple long-term steady states*, and this can be thought of as analogous to *multiple equilibria* in the Arrow model. The main distinction of our work is the underlying mechanism of discrimination; importantly, our work does not rely on applicants to make any decisions. The results of Arrow (and extensions such as Coate and Loury (1993), Moro and Norman (2004), Levin (2009), etc.) rely on coordination failure as the source of discrimination, where applicants from different groups make different skill investment decisions. Contrastingly, our work shows that a similar phenomenon of two ex-ante identical groups ending up in two distinct outcomes can occur, without relying on any decisions made by applicants. Another distinction is the difference in the technical definition of a "steady state" (this paper) and an "equilibrium" (Arrow model), though conceptually we think of these as similar ideas.

Dynamic models. There are papers that study dynamic models of statistical discrimination, where the dynamics considered by these papers model a fundamentally different idea than the dynamics that we study. Blume (2006) and Levin (2009) develop dynamic models based on Arrow (1973) where firms maintain beliefs over the worker's skill, which results from costly investments. Fryer Jr (2007) and Bardhi et al. (2020) study the dynamics of worker's employment over their career. Bolte et al. (2020) develop a dynamic referral-based hiring model where homophily induces persistent disparities.

Fairness in machine learning. Another closely related area is the recent but large literature on fairness in machine learning; see Barocas et al. (2023), Chouldechova and Roth (2020), and Mehrabi et al. (2021) for an overview of this literature. Our model studies a firm's decision of *whether* to hire an applicant or not, which is akin to the classification problem in machine learning. This is in contrast to most of the models of statistical discrimination, where the decision is related to setting wages. The "classical" definitions of fairness in machine learning are often motivated by the hiring or school admissions applications (e.g., Barocas et al. (2023), Chouldechova and Roth (2020), Mehrabi et al. (2021)). Within our model, one can view such fairness definitions as a short-term intervention. There is also a collection of papers that bring up critiques of the classical fairness definitions (e.g., Bao et al. (2021), Kasy and Abebe (2021), Corbett-Davies et al. (2023)) as well as works that posit economic-driven frameworks for fair decision-making (e.g., Kleinberg et al. (2018), Chohlas-Wood et al. (2021)) — our work has a similar motivation to this stream of literature.

More directly related to our work are papers that study a dynamic setting and evaluate long-term outcomes. Liu et al. (2018) show that even in a one-step feedback model, imposing a fairness constraint does not necessarily lead to an improvement and can even potentially cause harm. Hashimoto et al. (2018) study a dynamic model in which those who experience a high error are likely to leave the system, and they show that repeated empirical risk minimization can lead to exacerbated disparities over time. Liu et al. (2020) study a dynamic model where users invest effort as a best response to the current system, where they focus on how the lack of realizability can lead to an unfair equilibrium outcome. Fu et al. (2022) study how imposing an equal opportunity fairness constraint can reduce a firm's incentive to invest in better predictions, which can make all groups worse off.

Algorithmic hiring and operations. Our paper also relates to the growing operations literature that study models and algorithms for hiring or school admissions. Garg et al. (2020) study the impact of using standardized tests for college admissions when different groups of applicants have unequal access to the test. They extend the model of Phelps (1972) to multiple features, testing costs, and competition across schools. Farajollahzadeh et al. (2025) study a two-stage hiring model with information asymmetry across groups, where they show that the Rooney Rule intervention may fail to improve hiring outcomes, especially when firms compete to hire from a common pool. Several other papers study the effectiveness of interventions or mechanisms for hiring (Kleinberg and Raghavan 2018) and school admissions (Faenza et al. 2020, 2022). A line of work studies the impact of algorithmic monoculture, where multiple firms use a common algorithm to score applicants to make interview or hiring decisions (Kleinberg and Raghavan 2021, Peng and Garg 2024, Baek et al. 2025). Aminian et al. (2023) develop algorithms for equitable hiring in a sequential setting, and they show that when signals can be biased, incorporating fairness constraints can improve long-term outcomes. Salem and Gupta (2023) develop algorithms for sequential hiring where information about applicants are given as a partial order, modeling the idea that not all applicants are directly comparable. Purohit et al. (2019) and Epstein and Ma (2022) study the sequential hiring problem under uncertainty in offer acceptances.

The rest of the paper proceeds as follows. Section 2 presents our model. Section 3 characterizes the learning outcome and the impact of the initial noise and the group size on it as well as interventions. In Section 4, we extend our model to one where two groups compete for a limited number of spots. Section 5 contains simulations that mimic an ML-based system that makes hiring decisions over time. Section 6 provides several extensions and robustness checks for our main results, including an extension of the model where applicants can exert effort. Section 7 concludes, while the Appendix presents the omitted proofs from the text.

2. Model

We consider a multi-period labor market of a firm that hires from an applicant pool. Each time period $t = 1, 2, \dots$ represents a *different* pool of applicants, whereas the firm stays the same across periods. For example, time periods can be years, and the applicant pool can be those who are graduating college that year. The applicants are divided into mutually exclusive *groups* (e.g., graduates from different colleges), and the firm interacts with each group separately. We describe the dynamics of the interactions between the firm and *one group* of applicants. In Section 4, we consider a variant of the model where two groups are competing for a limited number of positions.

A group at one time period is represented by a continuum of applicants with a total mass of $P > 0$, which we refer to as the *group size*. Each applicant has a scalar *skill*, and the distribution of the skill is $\mathcal{N}(0,1)$. Let S_t be a random variable denoting the skill of an applicant drawn at random at time t.

Observation and hiring rule. At time t, the firm receives an observation O_t for each applicant, which represents a noisy signal of the applicant's true skill S_t. At each time period t, there is a *noise variance* σ_t^2 associated with the group, where a lower noise variance corresponds to a higher evaluation ability by the firm. The firm observes $O_t = S_t + \epsilon_t$ where $\epsilon_t \sim \mathcal{N}(0, \sigma_t^2)$ is drawn independently for each applicant. The firm then computes their belief on the expected value of the applicant's skill, $\mathbb{E}[S_t \mid O_t]$, which we refer to as the *inferred skill*. We assume the firm is aware of both the prior distribution of skill as well as the noise variance, hence the inferred skill has the form:

$$\mathbb{E}[S_t \mid O_t] = \frac{O_t}{1+\sigma_t^2}.$$

Then, the firm hires the applicant if and only if $\mathbb{E}[S_t \mid O_t] > \tau$, where $\tau > 0$ is an exogenous parameter. We define the *hiring rate* at time t to be $q_t = \Pr(\mathbb{E}[S_t \mid O_t] > \tau)$, the percentage of applicants who are hired.

Noise dynamics. We assume that the noise variance for the next time step updates as

$$\sigma_{t+1}^2 = \frac{1}{(P \cdot q_t)^b}, \tag{1}$$

for some $b \in \mathbb{R}_+$. That is, the variance is inversely related to the mass of applicants hired at the previous time step, $P \cdot q_t$. We discuss this modeling assumption in Section 2.1.

Initialization and steady states. Given parameters P and b, a process is initialized with the initial hiring rate $q_0 \in [0,1]$, which defines the noise variance at the first time step: $\sigma_1^2 = 1/(Pq_0)^b$. If the hiring rate for a group converges to a limit, $\lim_{t\to\infty} q_t = q_\infty$, then we say that q_∞ is a *steady state*. We say that q_∞ is a *stable* steady state if there exists an $\epsilon > 0$ such that if $q_0 \in [q_\infty - \epsilon, q_\infty + \epsilon]$, the process converges to q_∞. Otherwise, it is an *unstable* steady state. We say that q_∞ is an *active* steady state if $q_\infty > 0$, and we say that it is *inactive* if $q_\infty = 0$.

Remark 1 (Multiple groups). The model above specifies the dynamics of a single group. If there are multiple groups, we assume that a firm interacts with each group separately, and hence, the above process runs independently for each group. Since the firm simply hires everyone whose expected skill is above the threshold τ, there is no interaction or competition between groups. In Section 4, we consider an extension of this model where two groups are competing for a limited number of positions at a single firm, which captures the interaction between two groups.

2.1. Discussion of Dynamics

The dynamics on the noise variance of the observation represent the salient modeling feature of our work. The noise variance decreasing in the previous mass of hires formalizes the idea that a firm is better at evaluating applicants from a group if they have hired more from that group in the past. We describe two motivating reasons for this modeling decision. First, once an applicant is hired and becomes an employee, the firm observes their performance and learns about their true skill. Therefore, the past hires serve as labeled data points that the firm can learn from to evaluate future applicants (as is also done in Li et al. (2020) and Komiyama and Noda (2024)). Second, past hires become employees of the firm who often take part in the hiring process, and they will be better at evaluating those who have a similar background to them (e.g., alumni of a particular school have a better sense of what a "good" GPA from that school is). Both of these mechanisms lead to the phenomenon that the firm's ability to evaluate applicants improves in the number of hires from their group.

Relatedly, another key assumption of our model is that the noise variance is a function of the hiring decisions from a *finite* number of previous time steps, rather than *all* previous time steps. Note that for both of the aforementioned motivating reasons, hiring an applicant will not improve the firm's evaluation ability *indefinitely*. This is because the population and the set of desirable skills are constantly changing, and hence observations from a long time ago are not predictive about today's applicants. Moreover, employees who take part in the hiring process do not stay at the firm forever. For simplicity, we model the noise variance as a function of the number of applicants hired from that group in the immediately previous time step. In Section 6.2, we consider an extension where we let the variance be a function of more than one previous round, and we show that the same results hold.

With respect to the exact functional form of the noise variance update, Eq. (1) is motivated by the idea that the variance of statistical estimators scales at a rate of $O(1/n)$ (e.g., central limit theorem), where n is the number of samples. In Section 5, we show that our findings continue to hold more broadly by conducting a simulation study that mimics a practical hiring system and avoids making any explicit modeling assumptions about the noise update rule. In Subsection 6.4, we consider a different functional form for the noise variance update rule, which is motivated by by a model wherein which a firm observes multiple features for each applicant,and the firm aims to learn which feature is informative of the applicant's skill. We show that our findings are robust and hold under this extension.

3. Steady State Convergence Analysis

In this section, we characterize the set of stable steady states, as well as which steady state arises as a function of the initial state.

At time t, given that the inferred skill is given by $\mathbb{E}[S_t \mid O_t] = O_t/(1+\sigma_t^2)$, the distribution of inferred skill over the population is given by

$$\mathbb{E}[S_t \mid O_t] \sim \mathcal{N}\left(0, \frac{1}{1+\sigma_t^2}\right).$$

Therefore, the proportion of applicants hired at time t is

$$q(\tau;\sigma_t^2) \triangleq \Pr\left(\mathcal{N}(0,1) > \tau\sqrt{1+\sigma_t^2}\right).$$

We characterize the dynamics via a function f which maps the noise variance at one step, σ_t^2, to the noise variance at the next step, σ_{t+1}^2. Specifically, let us define the function

$$x \mapsto f(x;P,\tau) \triangleq \frac{1}{(P\cdot q(\tau;x))^b}$$

for $x \geq 0$ that captures the mapping from a noise variance to the noise variance at the next time step. This function depends on the group size P and the threshold τ. The map f represents one step of the process, and hence the entire process is fully specified by repeatedly applying the function f. Hence, the function f specifying the one-step dynamics can be written as

$$f(x;P,\tau) = \frac{1}{\left(P\cdot(1-\Phi(\tau\sqrt{1+x}))\right)^b},$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

If $f(\sigma^2;P,\tau) > \sigma^2$, then the noise variance at the next time step is *worse* (larger) than the current time step. Conversely, if $f(\sigma^2;P,\tau) < \sigma^2$, then the noise variance at the next time step is better (smaller) than the current time step. Therefore, understanding the convergence dynamics reduces to understanding how the function $f(x;P,\tau)$ compares to the function $y = x$. Figure 1 plots an example of the function $f(x;P,\tau)$ as well as $y = x$. The next lemma characterizes the number of intersections of $f(x;P,\tau)$ and $y = x$.

LEMMA 1. *For any P and τ, the function $f(x;P,\tau)$ for $x \geq 0$ intersects with $y = x$ in at most two points.*

We note that the function $f(x;P,\tau)$ is not convex in x, which would have been a sufficient condition to prove the above lemma. To prove this result, we equivalently show that the function $v(x) = x/f(x;P,\tau)$ intersects the line $y = 1$ at most twice, which we establish by showing that $v(x)$ has at most one stationary point. The proof of this result (and all other results) can be found in the appendix.

Using this, our main result characterizes the learning dynamics and the set of stable steady states.

THEOREM 1. *For any given P, there exists $\bar{\tau} \in \mathbb{R}_+$ such that:*

- *If $\tau > \bar{\tau}$, then there exists only one stable steady state which is inactive.*
- *If $\tau < \bar{\tau}$, then there exists exactly one active stable steady state and one inactive steady state.*

Furthermore, for a given $\tau < \bar{\tau}$, there exists a variance threshold $\bar{\sigma}^2$ such that any noise variance $\sigma^2 < \bar{\sigma}^2$ converges to the active stable steady state, and any noise variance $\sigma^2 > \bar{\sigma}^2$ converges to the inactive state.

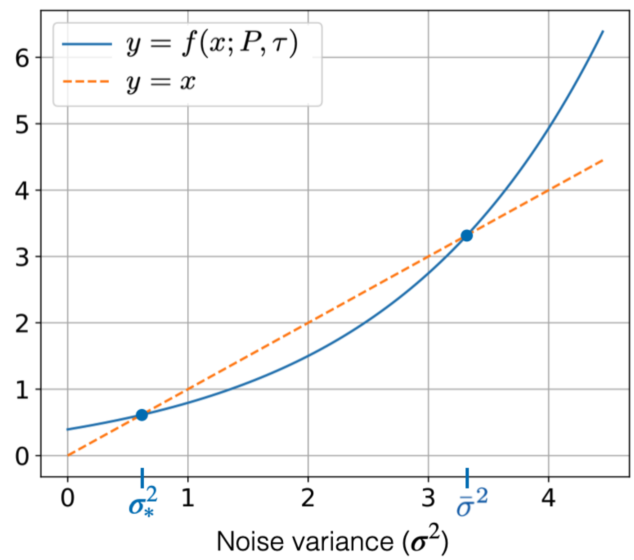


Figure 1 **Function $f(x;P,\tau)$ for $P = 12$, $\tau = 1$, and $b = 1$. The first intersection of f with $y = x$ represents the active state, marked as σ_*^2. The second intersection corresponds to the threshold $\bar{\sigma}^2$ in Theorem 1: if the initial noise variance is below this value, it converges to the active state, whereas if it is above the threshold, it converges to the inactive state.**

Theorem 1 characterizes the set of steady states as well as the conditions on convergence to each one. This result establishes that the inactive steady state always exists, and there is at most one active stable steady state. Since these are unique, from now on, we will refer to these in the shortened form as *the inactive state* and *the active state*, where the latter may not exist.

We let $\bar{\sigma}^2(P,\tau)$ denote the variance threshold corresponding to the second bullet of Theorem 1, where we define $\bar{\sigma}^2(P,\tau) = 0$ if the active state does not exist. In Figure 1, the larger intersection of $f(x;P,\tau)$ and $y = x$ is $\bar{\sigma}^2(P,\tau)$. Similarly, we let $\sigma_*^2(P,\tau)$ denote the noise variance at the active state, where we define $\sigma_*^2(P,\tau) = \infty$ if the active state does not exist. In Figure 1, the smaller intersection of $f(x;P,\tau)$ and $y = x$ is $\sigma_*^2(P,\tau)$. With this notation in hand, it is worth highlighting the dynamics of the noise variance evolution. For any initial noise variance $\sigma^2 \le \sigma_*^2(P,\tau)$, the noise variance increases over time and converges to the active state $\sigma_*^2(P,\tau)$. For any noise variance $\sigma^2 \in (\sigma_*^2(P,\tau), \bar{\sigma}^2(P,\tau))$, the noise variance decreases over time and again converges to the active state $\sigma_*^2(P,\tau)$. In contrast, for any noise variance $\sigma^2 > \bar{\sigma}^2(P,\tau)$, the noise variance increases over time and converges to the inactive state. We note that the second intersection, $\bar{\sigma}^2(P,\tau)$ is an additional active steady state, but it is not *stable*.

We now describe several practical implications and corollaries of Theorem 1.

3.1. Impact of the Initial Noise Variance

Taking $\tau < \bar{\tau}$ as fixed, Theorem 1 formalizes the importance of the *initial* noise variance. The eventual steady state of a group is simply determined by the threshold $\bar{\sigma}^2$, where a group will converge to the active state if and only if their initial noise variance is below the threshold. Then, comparing two otherwise identical groups, even a small difference in their initial states (e.g., just above and below $\bar{\sigma}^2$) can lead to two completely different long-term prospects, despite their underlying true skill distribution being identical. Put differently, a group starting with even a small disadvantage can lead to greatly exacerbated disparities in the long run.

3.2. Impact of Group Size

Next, we evaluate the impact of the group size on the resulting steady state. We show that belonging to a group with a smaller size can be disadvantageous in two ways.

Proposition 1. *Fix $\tau > 0$.*

- *The noise variance threshold to converge to the active state, $\bar{\sigma}^2(P,\tau)$, is increasing in P.*
- *The proportion hired at the active steady state, $q(\tau, \sigma_*^2(P,\tau)) \in [0,1]$, is increasing in P.*

First, the threshold that the noise variance needs to be below in order to converge to the active state is higher for a larger group. Therefore, even if two groups start with the same noise variance, it may be that the larger group converges to the active state while the smaller group converges to the inactive state. Second, even if two groups are in the active state, the *proportion* of those who are hired (which normalizes for group size) is smaller for the smaller group.

3.3. Interventions: Need for Drastic Measures

It is natural to consider interventions that help a group converge to the active state. Theorem 1 specifies what this intervention needs to achieve: the noise variance associated with evaluating that group must fall below a critical threshold $\bar{\sigma}^2(P,\tau)$. This threshold marks the point at which the feedback loop begins to positively reinforce itself, enabling sustained improvement in the firm's evaluation accuracy and long-term outcomes for the group. Crucially, interventions that fail to reduce the noise variance sufficiently will only have a temporary impact, as the system will revert to the inactive state once the intervention is removed.

The exact nature of the intervention can take many forms. For example, one can increase the number of applicants hired at one time step to decrease the noise in the next time step, as formalized in the next result.

Proposition 2. *If $\sigma^2_*(P,\tau) < \infty$, then there exists $q \in [0,1)$ such that the hiring q proportion of applicants from this group in a single round results in the process converging to the active state.*

Another possible approach is to directly decrease the noise by improving the evaluation process (e.g., by spending more effort on evaluation, conducting more interviews, etc.).

Regardless of the type of intervention, our results highlight the importance of *drastic* interventions. By significantly increasing the firm's hiring from a disadvantaged group or directly improving its evaluation process, such interventions can ensure that the noise variance crosses the critical threshold. Once this is achieved, the group can naturally converge to the active state, maintaining its improved status even after the intervention ends. In contrast, mild interventions, even those in place for a long time, that do not lower the noise variance below the threshold are ineffective in the long run. Such interventions might temporarily boost hiring while they are in effect but fail to create the self-reinforcing dynamics necessary for sustained improvement, causing the group to fall back to the inactive state when the intervention is lifted.

3.3.1. Demographic parity is sufficient. It is natural to ask: how drastic does an intervention have to be? We show that imposing hiring rates to be equal across groups, where the hiring rate matches that of a group in the active state, is a sufficient intervention. It will either induce convergence to the active state, or it may not, in which case this serves as evidence that there is no active state.

Specifically, suppose there are two groups, A and B, where group A is in the active steady state and group B is not. We consider the intervention of increasing the hiring rate of group B to match that of group A. Specifically, let $q_*(A) \in [0,1]$ be the hiring rate corresponding to the active steady state for group A. Then, we show that hiring $q_*(A)$ proportion of group B applicants will lead group B to the active state, if it exists.

Proposition 3. *Suppose at time t, the hiring rate of group B is set to be $q_*(A)$; i.e., $q_t(B) = q_*(A)$. Then, if the active state for group B exists, then $\sigma^2_{t+1}(B) < \bar{\sigma}^2(B)$, and group B will converge to the active state.*

Proposition 3 does not say that group B will *always* converge to an active steady state, as this may not exist for group B (which would occur if $P(B)$ is too small). However, if demographic parity does not work, this serves as evidence that the active steady state does not exist, and hence *no* intervention that decreases the noise variance can be successful. That is, even if one were to further increase the hiring rate for group B beyond demographic parity, it would not converge to an active state.

We note that setting explicit quotas for demographic groups is often not legal; for example, such quotas are prohibited in the United States. However, we do not interpret demographic parity as necessarily imposing such quotas. Rather, we view demographic parity as a benchmark: a property that firms may aim to achieve through broader recruitment efforts or structural changes to their hiring process. For instance, a firm might increase representation by expanding its recruiting pipeline to include different institutions or by partnering with organizations that reach underrepresented groups. Our result should be interpreted in this spirit: if a firm is able to increase hiring rates to satisfy demographic parity, then this is always sufficient to induce convergence to the favorable steady state, if one exists.

4. Competition Across Groups

So far we have assumed that the firm hires all applicants whose inferred skill is higher than τ, where $\tau > 0$ is an exogenous parameter. We now consider a variant of the model with competition across groups. We assume the firm has a total hiring capacity of $C > 0$, and they hire the top C mass of applicants across the groups with the highest inferred qualities. Given two groups A and B, this model can equivalently be stated as the firm selecting a time-dependent threshold τ_t such that

$$P(A)\Pr(\mathbb{E}[S_t(A) \mid O_t(A)] > \tau_t) + P(B)\Pr(\mathbb{E}[S_t(B) \mid O_t(B)] > \tau_t) = C,$$

and then hires everyone whose inferred skill is larger than τ_t. We normalize the size of group A to be 1, and that group A is the smaller group: $1 = P(A) \le P(B)$. We assume that the initial condition, $(q_0(A), q_0(B))$, satisfies $P(A)q_0(A) + P(B)q_0(B) = C$.

We first show that the hiring rates move monotonically and converge to a limit.

PROPOSITION 4. *For any t and any group g, if $q_t(g) > q_{t-1}(g)$, then $q_{t+1}(g) > q_t(g)$. If $q_t(g) < q_{t-1}(g)$, then $q_{t+1}(g) < q_t(g)$. If $q_t(g) = q_{t-1}(g)$, then $q_{t+1}(g) = q_t(g)$. Lastly, $q_t(g)$ converges to some limit $q_\infty(g) \ge 0$ as $t \to \infty$.*

4.1. Characterizing Steady States

We say that $(q_\infty(A), q_\infty(B))$ is an *inclusive* steady state if both $q_\infty(A) > 0$ and $q_\infty(B) > 0$. If one of $q_\infty(A), q_\infty(B)$ is positive and the other is 0, then we say that the steady state is *exclusive*.

We first characterize two conditions that result in an exclusive steady state. The first condition is that the initial hiring rate for one group is sufficiently small.

THEOREM 2. *For any group g and any $C \in (0, 1/2)$, there exists a $\tilde{q} > 0$ such that if $q_0(g) < \tilde{q}$, then $\lim_{t\to\infty} q_t(g) = 0$.*

Theorem 2 implies that when the discrepancy between initial hiring rates is sufficiently large and one group initially has a small hiring rate, then the firm will eventually exclusively hire from the other group. Note that this result does not depend on the size of the groups; the hiring rate for the larger group can go to 0 if their initial hiring rate is sufficiently small. Theorem 2 implies that as long as $C < 1/2$, there always exist initial conditions that result in $q_\infty(A) = 0$, and there exist initial conditions that result in $q_\infty(B) = 0$.

Next, we show that even if the initial discrepancy in hiring rates is small, an exclusive steady state arises when the total hiring capacity C is sufficiently small.

THEOREM 3. *Suppose either of the following two conditions hold:*

- $q_0(A) < q_0(B)$, *or*
- $q_0(A) = q_0(B)$ *and* $P(A) < P(B)$.

Then, there exists a C' *such that if* $C < C'$, $q_t(A) \to 0$ *and* $q_t(B) \to C$ *as* $t \to \infty$.

Theorem 3 specifies two conditions in which the hiring rate of group A goes to 0 when C is sufficiently small. The first condition is when the initial hiring rate of group A is smaller than group B. The second condition is when the initial hiring rates are the same, but the size of group A is strictly smaller than that of group B. Therefore, both the initial hiring rate and the group size plays a crucial role in the system dynamics.

Note that the capacity C is comparable to the parameter τ in the original model, which both represents the "selectivity" of the firm. Specifically, a larger value of τ decreases the hiring rate, which is analogous to a larger value of C. Therefore, Theorem 3 is analogous to Theorem 1, which states that if τ is large enough, there is no active state and the group converges to an inactive state. In the competition model, when C is small enough, the hiring rate for the disadvantaged group goes to 0.

On the positive side, we show that an inclusive steady state can arise when C is large enough, and the initial discrepancy in hiring rates is small. We first show this when group sizes are equal, where there is an inclusive steady state where the hiring rate for both groups is exactly $C/2$.

THEOREM 4. *Suppose* $P(A) = P(B) = 1$. *There exists* $C' < 1$ *such that if* $C \in (C', 1)$, *there exists* $\delta > 0$ *such that if* $|q_0(A) - q_0(B)| < \delta$, *then* $q_\infty(A) = q_\infty(B) = C/2$.

Next, we show a similar result when the group sizes differ. We show that an inclusive steady state arises when group sizes slightly differ and the initial discrepancy in hiring rates is small. However, this inclusive steady state does not have equal hiring rates — the hiring rate at the steady state is smaller for the smaller group.

THEOREM 5. *Suppose* $P(B) > P(A) = 1$. *There exists a* $C' < (P(A) + P(B))/2$ *such that if* $C \in (C', (P(A)+P(B))/2)$, *there is a steady state* $(q_\infty(A), q_\infty(B))$ *where* $0 < q_\infty(A) < q_\infty(B)$. *Moreover, there exists a* $\delta > 0$ *such that if* $q_0(A) > q_\infty(A) - \delta$, *then the process converges to* $(q_\infty(A), q_\infty(B))$.

In summary, under the competition model, there are multiple distinct steady states that differ in long-term hiring rates for each group. As in the original model, which of these steady state arises is crucially determined by the firm's initial evaluation ability for each group.

4.2. Increasing Population Size Exacerbates Disparities

We show that when the population size increases over time, this worsens the gap in hiring rates between two groups compared to when the population size stays constant. For this extension, we consider just two time steps: $s-1$ and s, and we evaluate what happens when the group size increases at time s. We denote the size of group g at time t as $P_t(g)$. We compare the outcome at time s under two settings: (1) group size stays the same for both groups and (2) group size increases for both groups, with the same multiplicative factor. Let $P_s^{(n)}(g)$ be the size of group g at time s under setting (n). Then, setting (1) corresponds to $P_s^{(1)}(A) = P_{s-1}(A)$, $P_s^{(1)}(B) = P_{s-1}(B)$, while setting (2) corresponds to $P_s^{(2)}(A) = \beta P_{s-1}(A)$, $P_s^{(2)}(B) = \beta P_{s-1}(B)$, for $\beta > 1$.

We assume that the only difference between settings (1) and (2) is the population increase at time s; hence the noise variance at the beginning of time s is the same under both (1) and (2). Then, at time s, the threshold τ_s changes across the two settings; it will be larger under (2) since the population increases, but the total mass hired stays the same. This induces the hiring rate of each group at time s to differ. For group $g \in \{A, B\}$, let $q_s^{(n)}(g)$ be the hiring rate at time s for setting (n). We define $\zeta_s^{(n)}(g) = (P_s(g) q_s^{(n)}(g))/C$ to be the fraction of total hires that come from group g under setting (n). We show that the group with the lower noise variance is better off in setting (2) than setting (1) in terms of $\zeta_s^{(n)}(g)$.

Proposition 5. *If $\sigma_s^2(A) < \sigma_s^2(B)$, then $\zeta_s^{(1)}(A) < \zeta_s^{(2)}(A)$.*

This result states that if group A has a smaller noise variance, the hired pool will have a higher fraction of group A applicants when the population increases, compared to when the population stays the same. Therefore, increasing the population, which increases the level of competition, favors the group with the smaller noise variance.

5. Simulation: ML-based Hiring System

To complement the theoretical analysis, we conduct a simulation study in which the firm uses standard machine learning tools to estimate applicant quality based on observed data. These simulations aim to mimic a practical implementation of a hiring system that is deployed over a time frame, and our simulation results show that this system exhibits the same qualitative behavior established by our theoretical results of the stylized model.

5.1. Simulation Setup

A firm evaluates P applicants per period from the same group, where applicant i is represented by a feature vector $X_i \sim \mathcal{N}(0, I_d)$, with $d = 10$. The applicant's true skill is given by $S_i = \langle \beta, X_i \rangle + \epsilon_i$, where $\epsilon_i \sim \mathcal{N}(0, 1)$ and β is an unknown parameter. At time t, the firm uses an estimator $\hat{\beta}_{t-1}$ to compute a score $\hat{S}_i = \langle \hat{\beta}_{t-1}, X_i \rangle$ for each applicant i. The firm hires everyone whose score $\hat{S}_i$ exceeds a fixed threshold τ . The threshold τ is determined through a *target hiring rate* $\tilde{q} \in [0,1]$. Given $\tilde{q}$, τ is chosen so that if β were known perfectly, then the hiring rate would equal $\tilde{q}$.

For each hired applicant at time t, the true skill S is revealed. The firm uses the data of hired applicants at time t to estimate $\hat{\beta}_t$ using linear regression with regularization. Specifically, at time t, suppose the firm hires n applicants, and let $\mathbf{X} \in \mathbb{R}^{n\times d}$ and $\mathbf{S} \in \mathbb{R}^n$ represent the dataset corresponding to those applicant's features and true skills respectively. Then, the estimator $\hat{\beta}_t$ is computed as $\hat{\beta}_t = \left(\lambda\mathbf{I} + \mathbf{X}^\top\mathbf{X}\right)^{-1}\left(\mathbf{X}^\top\mathbf{S}\right)$. In our simulations, we set $\lambda = 10^6$. This estimator corresponds to linear regression with an ℓ_2 regularization, which is also equivalent to Bayesian linear regression.

To initialize $\hat{\beta}_0$, the firm starts with an initial dataset of size M. That is, M applicant features are drawn at random, and their true skill is known, and the firm estimates $\hat{\beta}_0$ using this dataset. The parameter M is representative of the firm's initial evaluation ability, since a larger value of M implies a better initial estimate of β.

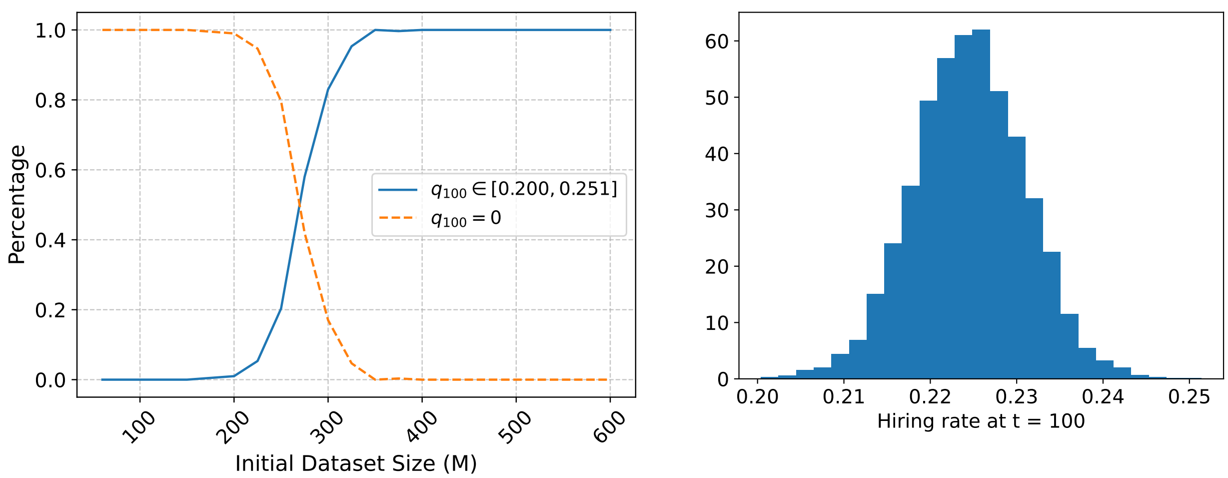
We run simulations by varying both $\tilde{q} \in \{0.15, 0.2, 0.25, 0.3, 0.35, 0.4\}$ and $M \in [50, 2000]$, fixing $P = 5000$. For each set of parameters, we simulate the process over $t = 100$ time steps, and we repeat this 300 times. The true parameter β was drawn from $N(0,\mathbf{I})$. We are interested in q_{100}, the hiring rate at $t = 100$. Figure 2 plots q_{100}, averaged over 300 runs, for each value of $\tilde{q}$ and M.

Figure 2 For each value of M and target hiring rate ($\tilde{q}$), the plot shows the hiring rate at time $t = 100$, averaged over 300 runs. The population P is fixed to 5000.

5.2. Results

The simulation results mirror several findings that are consistent with the theoretical results.

Two steady states. Figure 2 reveals a sharp transition in the long-term hiring rate as a function of M. For each value of $\tilde{q}$, the hiring rate is consistently zero when M is small. As M increases, there is a sudden increase in the hiring rate, which stabilizes to a constant number. This pattern mirrors the bifurcation behavior established in Theorem 1: the hiring rate converges either to an inactive state with no hiring or to an active state with a stable positive hiring rate, and the outcome depends on whether the initial estimation ability is above or below a threshold.



(a) The percentage of simulation runs that end in each steady state as a function of the initial dataset size M.

(b) The distribution of q_{100}, conditioned on $q_{100} > 0$. The mean is 0.224 and the standard deviation is 0.0065.

Figure 3 **For $\tilde{q} = 300$, we plot (a) the percentage of simulation runs that end in each steady state, and (b) the distribution of the hiring rate at time $t = 100$ when the hiring rate is non-zero.**

Next, we evaluate the distribution of outcomes, rather than solely looking at average statistics. We fix $\tilde{q} = 0.3$, and we observe that for any value of M, the hiring rate at $t = 100$ can be bifurcated into exactly two outcomes: either the hiring rate is 0, or the hiring rate is between 20.0% and 25.1%. The two outcomes can be thought of as the *inactive* and *active* steady states respectively; because the process is random, the active steady state is not represented by one number. In Figure 3, we plot (a) the percentage of simulation runs that end in each steady state, and (b) the distribution of the hiring rate at time $t = 100$ when the hiring rate is non-zero.

Figure 3a confirms that the long-run hiring rate exhibits only two possible outcomes: either zero or to a non-zero range between approximately 20% and 25%. This reinforces the existence of distinct inactive and active steady states. Figure 3b shows that even within the active steady state, the long-run hiring rates cluster tightly around 22.4%, with variance arising from the stochasticity in the learning process. This concentration highlights that while the process stabilizes, residual randomness still influences the precise hiring rate attained. These results demonstrate that the key insights of the theoretical model, the existence of two steady states, and the dependence on the initial evaluation ability emerge in a realistic ML-based environment.

We note that a key driver of these results is the regularization used in estimating the parameter β, which is equivalent to assuming a Bayesian model where the prior on β is centered at 0. This regularization shrinks the estimator $\hat{\beta}$ toward zero, particularly when the amount of available data is small. As a result, if the initial dataset is too limited, the estimated $\hat{\beta}$ may be close to zero, causing the firm to underestimate applicant quality and hire no one.

Observed hiring rate below target. Interestingly, Figure 2 shows that the hiring rate is *consistently below* the target hiring rate $\tilde{q}$. Recall that $\tilde{q}$ is the hiring rate that one should expect if β was known perfectly. This gap indicates that the algorithm does not perfectly recover the true parameter β; rather, the estimation retains a nontrivial amount of noise. As a result, the process converges to a steady state where the hiring rate reflects this residual noise. This mirrors Theorem 1, which establishes that even in the active steady state, the noise variance remains strictly positive, and thus the hiring rate is lower than what would be achieved under perfect information. In both the model and the simulation, the long-run hiring rate is endogenously determined by the amount of signal noise required to sustain that level of hiring.

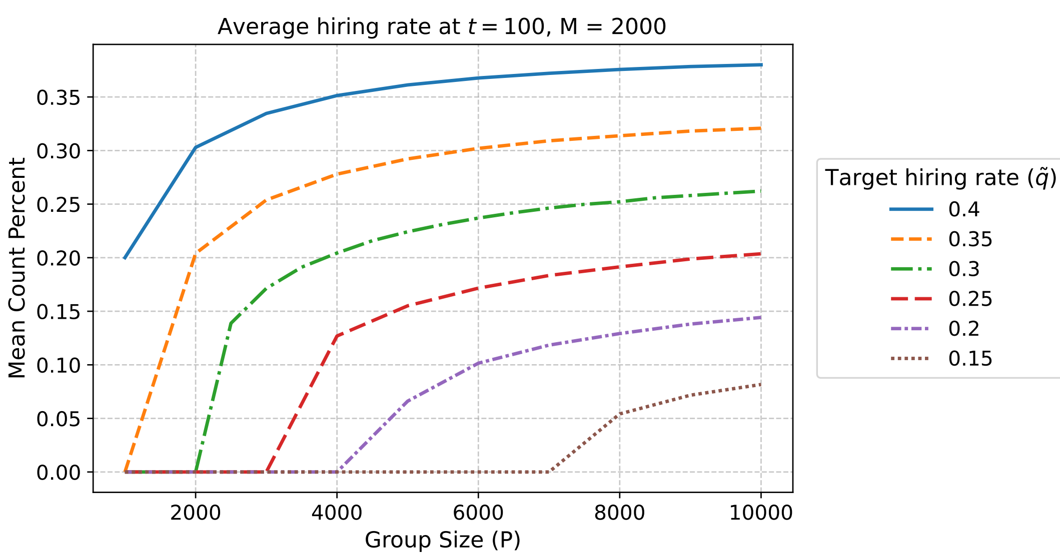


Figure 4 **The plot shows the hiring rate at time $t = 100$, averaged over 300 runs, where M is fixed at 2000 and the population size P is varied.**

Smaller groups are worse off even in the active state. Next, we vary the population size P, and we fix $M = 2000$, chosen to be large enough that the process should converge to active steady state, if it exists. Figure 4 plots the hiring rate q_{100} as a function of P, for various target hiring rates. Consistent with the result that smaller groups are worse off (Proposition 1), we see that the hiring rate at the active state increases with P. When P is small enough, the hiring rate is 0, which implies that there is no active state when the group size is too small.

6. Extensions

This section contains several extensions of the original model. In Section 6.1, we consider a model where applicants can improve their skills by exerting effort. In Section 6.2, we consider a variant of the model in which the noise variance can depend on more than one previous time step. In Section 6.3, we impose a maximum value that the noise variance can take. In Section 6.4, we consider a different functional form of the noise update rule that is motivated by a learning dynamic where a firm observes multiple features for each applicant and aims to learn which feature is informative.

6.1. What if applicants can improve their skill?

In this section, we study an extension where applicants can increase their skills by exerting effort, which incurs a cost. Examples of what this may entail are: receiving education and training, obtaining certifications and licenses, and gaining work experience.

At each time step, an applicant's skill is $S_t = \theta_t + \eta_t$, where $\theta_t \sim \mathcal{N}(0,1)$ is their *ability*, and $\eta_t \geq 0$ is their effort level. The firm observes $O_t = \theta_t + \epsilon_t$ and η_t, where $\epsilon_t \sim \mathcal{N}(0, \sigma_t^2)$ is drawn i.i.d. for each applicant. As before, the firm hires applicants whose expected skill is higher than the threshold $\tau > 0$; i.e., $\mathbb{E}[S_t \mid O_t, \eta_t] \geq \tau$. We assume the effort level is observed by the firm through, for example, the applicant's resume. We assume that the level of effort η_t is equal for all applicants in the same group. This is equivalent to each applicant choosing an effort level η_t being knowing the realization of their ability θ_t, but with knowledge of the noise variance σ_t^2.

We assume the cost of effort is quadratic, given by $c(\eta) = \frac{a}{2}\eta^2$ for some parameter $a \in \mathbb{R}_+$. The applicants choose the effort level $\eta^*(\sigma_t^2)$ that maximize the probability that they are hired, minus the cost of exerting the effort. We assume that $c(\tau) > 1$ (or equivalently, $a > 2/\tau^2$), which implies that a group of applicants will not benefit from the noise variance diverging to ∞.[3]

We establish that our main results continue to hold in this setting as well.

THEOREM 6. *There exists $\tilde{a} \in \mathbb{R}_+$ such that for any $a \geq \tilde{a}$ there exists $\tilde{\tau} \in \mathbb{R}_+$ for which:*

- *If $\tau > \tilde{\tau}$, then there exists only one steady state which is inactive.*
- *If $\tau < \tilde{\tau}$, then there is exactly one active stable steady state, and the inactive steady state exists.*

Furthermore, there exists $\tilde{\sigma}^2$ such that all noise variances $\sigma^2 < \tilde{\sigma}^2$ converge to the active state, and all noise variances $\sigma^2 > \tilde{\sigma}^2$ converge to the inactive state.

[3] Note that as $\sigma^2 \to \infty$, the firm's evaluation of each applicant's ability, $\mathbb{E}[\theta \mid \theta + \epsilon]$, goes to 0, regardless of the observation. If the cost of putting in effort $\eta = \tau$ is smaller than 1, then as the noise approaches ∞, the optimal action for all applicants is to exert an effort of τ, and every applicant will be hired. We assume that $c(\tau) > 1$ to avoid this behavior.

Theorem 6 is the analogue of Theorem 1 in a setting where the applicants can exert effort. The role of the assumption $a \geq \tilde{a}$ is for technical reasons to ensure that the optimal effort is interior and that the optimal effort does not drastically change when the noise variance changes.

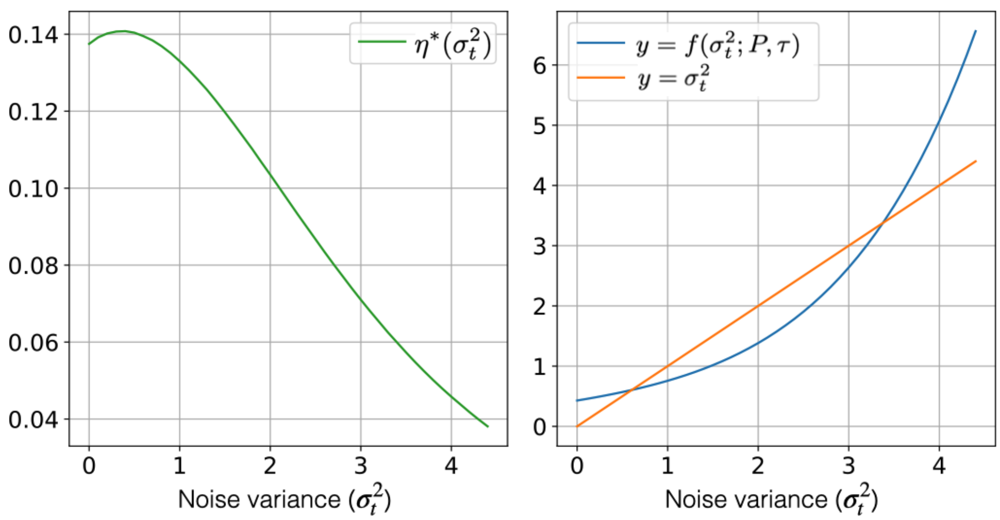


Figure 5 **For parameters $P = 12$, $\tau = 1$, $b = 1$, and $a = 2$, we plot the optimal effort and f as a function of the current noise variance. The left plots the optimal effort, $\eta^*(x)$ while the right plots the corresponding function $f(x; P, \tau)$ along with $y = x$ (analogous to Figure 1).**

Next, we show that the two steady states result in different levels of effort exerted, which then produces a discrepancy in the underlying skill distributions across these two states.

Proposition 6. *For any instance where $\sigma_*^2 < \infty$, the optimal effort at the active state is non-zero; i.e., $\eta^*(\sigma_*^2) > 0$. The optimal effort tends to zero when the noise goes to infinity; i.e., $\lim_{\sigma^2 \to \infty} \eta^*(\sigma^2) = 0$.*

Proposition 6 states that if groups A and B end up in the active and inactive state, respectively, applicants in group A will exert non-zero effort, while applicants in group B will exert zero effort. Then, even if the groups had identical ability distributions (i.e., θ), the skill distribution (i.e., $\theta + \eta$) for group A will be *higher* than (first-order stochastically dominate) group B. That is, the dynamics of our model can result in *tangible* difference in skills across the groups. This is in contrast to the baseline model (without effort), where even when two groups end up in two different steady states, the true skill distributions are still identical across groups, and the groups only differed in the noise variance *perceived* by the firm. This implication is of a similar flavor to the models of Arrow (1973) and Coate and Loury (1993), where there are multiple equilibria that differ in whether applicants invest in improving their skill.

6.2. What If the Noise Variance Depends on Multiple Previous Rounds?

In our baseline model, we assumed that the noise variance in the next round was solely a function of the current round's hiring decision. Here, we assume that the noise variance in the next round is a combination of the hiring decision in this round as well as the current noise variance. In particular, we assume that the noise variance for the next time step updates as

$$\sigma_{t+1}^2 = \alpha \frac{1}{\big(P \cdot q(\tau; \sigma_t^2)\big)^b} + (1-\alpha)\sigma_t^2,$$

where $\alpha \in (0,1]$ captures the degree of interpolation between the current hiring decision and the current noise variance (which in turn depends on hiring in previous rounds). Note that our baseline model corresponds to $\alpha = 1$. We establish our main result continues to hold for any α, and moreover, the critical thresholds for the noise variance convergence do not change with α.

Proposition 7. *For any given P and any $\alpha \in (0,1]$, the statements of Theorem 1 hold with the same quantities $\bar{\tau}$ and $\bar{\sigma}^2$.*

6.3. Bounding the Maximum Noise Variance

In our model, the dynamics of the noise variance was defined so that the variance can be unbounded. Specifically, in (1), if the proportion of applicants hired is zero ($q_t = 0$), then the noise variance at the next time step is infinity. In this section, we consider a modification to the model where the noise variance is bounded at a maximum value, for instance, because of transfer learning from other groups.

First, fix an instance of the original model where the active state exists. For this instance, let $\sigma_*^2 < \infty$ be the noise variance at the active steady state, and let $\bar{\sigma}^2$ be the noise threshold to converge to the active steady state, from Theorem 1.

Now, consider a modified instance where the noise variance is capped at a maximum value $M > 0$. Specifically, we assume the noise variance updates as

$$\sigma_{t+1}^2 = \min\left\{ \frac{1}{(P \cdot q_t)^b}, M \right\},$$

to replace the the transition in (1). We characterize the convergence of this modified instance, which depends on the value of M.

Proposition 8. *The set of stable steady states and the convergence properties are established in the following two cases.*

- *If $M < \bar{\sigma}^2$, then there is one stable steady state with noise variance $\min\{M, \sigma_*^2\}$. All noise variances converge to this steady state.*
- *If $M \geq \bar{\sigma}^2$, then there are two stable steady states, one at σ_*^2 and one at M. All noise variances $\sigma^2 < \bar{\sigma}^2$ converge to σ_*^2, and all noise variances $\sigma^2 > \bar{\sigma}^2$ converge to M.*

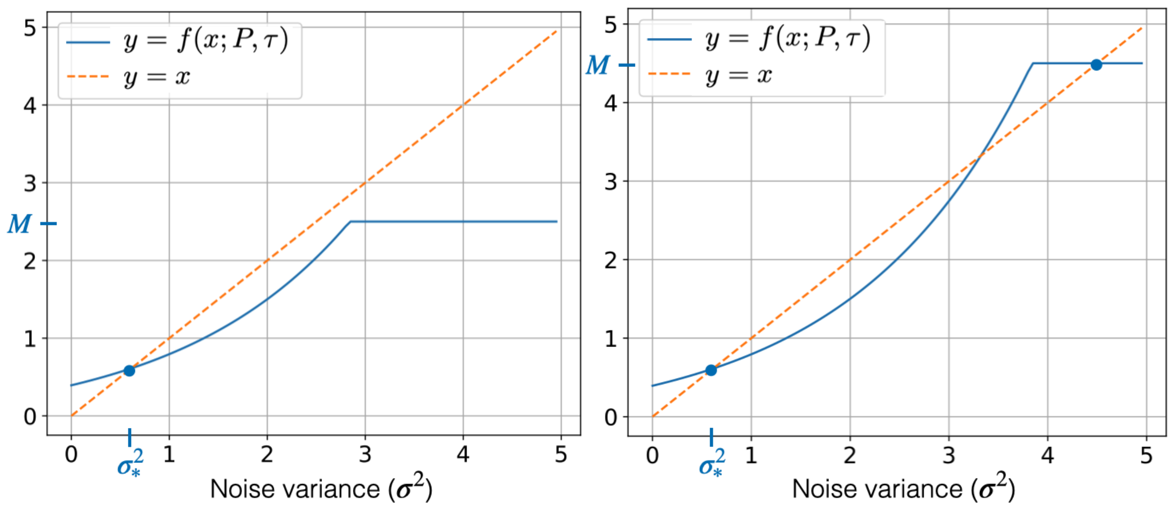


Figure 6 The noise transition function with a maximum noise variance of $M = 2.5$ and $M = 4.5$. The other instance parameters are the same as Figure 1: $P = 12$, $\tau = 1$, and $b = 1$. The blue dots refer to the stable steady states.

The two cases of this result can be seen pictorially in Figure 6. Because the noise variance is bounded, the inactive state does not exist (which requires the variance to diverge to ∞). However, in the second case of Proposition 8, there are still two distinct steady states that the system can converge to with noise variances σ_*^2 and M, respectively. These two steady states differ in the proportion of applicants who are hired at each of the steady states; specifically, the proportion hired is smaller at the steady state with noise variance M. Hence, in this scenario, one can interpret the steady state with variance M as the less desirable state, which a group converges to when the initial noise variance is higher than the threshold $\bar{\sigma}^2$.

6.4. Alternative Update Rule

We consider a different functional form of the noise update rule. As before, O_t represents a noisy signal of an applicant's true skill, S_t, and the firm computes the inferred skill, $\mathbb{E}[S_t \mid O_t]$, and hires the applicant if and only if $\mathbb{E}[S_t \mid O_t] > \tau$. We consider the dynamics in the distribution of the inferred skill, $\mathbb{E}[S_t \mid O_t]$. Specifically, we consider a model where inferred skill distribution takes the form:

$$\mathbb{E}[S_t \mid O_t] \sim \mathcal{N}\left(0, c_1 + \frac{c_2}{1 + b\, a^{P q_{t-1}}}\right), \tag{2}$$

where $c_1, c_2, b > 0$ and $a \in (0,1)$ are parameters. Note that the original model from Section 2 assumes that the inferred skill is distributed as

$$\mathbb{E}[S_t \mid O_t] \sim \mathcal{N}\left(0, \frac{1}{1 + 1/(P q_{t-1})^b}\right).$$

The update rule of (2) is motivated by considering the limit of a microfounded learning dynamic, where each applicant has a set of features, and a firm learns about which feature is informative. The details of this learning dynamics is developed in Appendix D.

We analyze the dynamics given by (2). Specifically, the parameters of the process are $c_1, c_2, b > 0$, $a \in (0,1)$, $\tau > 0$ and group size P. The process is initialized with a hiring rate $q_0 \in [0,1]$. Given this, we show that there are parameter regimes in which there are at least two distinct stable steady states.

THEOREM 7. *For any P and $\tau > 0$, there exists constants $B, D > 1, C > 0$ such that if $b \geq B$, $c_1 < C$, and $a \in (\exp(-b), b^{-D})$, then the dynamics specified by* (2) *has at least two distinct stable steady states. All steady states are active; $q_\infty > 0$.*

We denote by $g(x)$ to represent the mapping between the variance of the inferred skill from one time step to the next time step. Then, whether this variance increases or decreases in one step depends on whether $g(x)$ is greater than or larger than x. Similar to the function f from Figure 1, a steady state corresponds to a point where $g(x)$ intersects the line $y = x$. An example of an instance with exactly two stable steady states is displayed in Figure 7. The points marked x_1 and x_3 are stable steady states, while x_2 is unstable. Therefore, for this instance, the process converges to x_1 if the initial variance of the inferred skill is less than x_2. If the initial variance was larger than x_2, then the process converges to x_3.

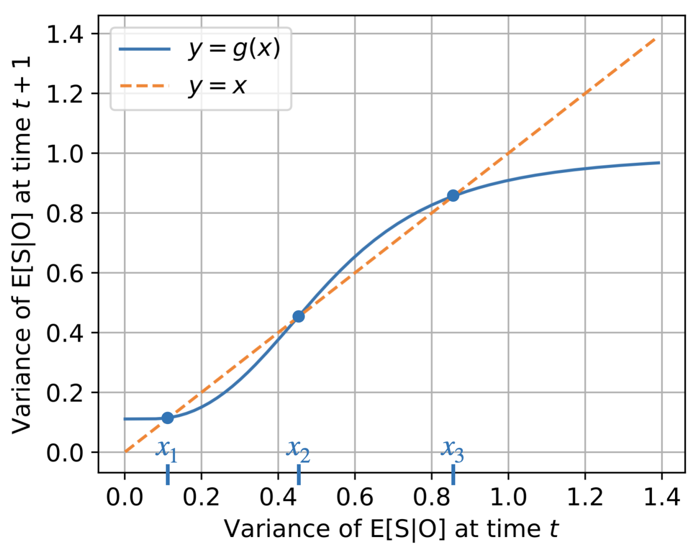


Figure 7 Example of the function $g(x)$ with $P = 1$, $\tau = 1$, $b = 10$, $a = 10^{-12}$, $c_1 = 0$, and $c_2 = 1$. x_1 and x_3 are stable steady states, while x_2 is an unstable steady state.

Under the definition of $g(x)$, a larger value of x corresponds to a "more desirable" steady state. Since the firm hires everyone with inferred skill larger than $\tau > 0$, a greater variance for the inferred skill implies that a greater portion of applicants will be hired. Therefore, in Figure 7, the steady state at x_3 is more desirable than x_1, in that the corresponding hiring rate is higher at x_3 than x_1. However, both steady states are active, meaning that the fraction of applicants hired is not zero. This is in contrast to the result of Theorem 1 for the model of Section 2, where there were two steady states, but one of them was inactive. However, the implications of the results are the same across the two models — there are multiple distinct steady states, and which one the process converges to depends on the initial state of the group.

7. Conclusion

In this paper, we considered a dynamic labor market model and established how disparities in hiring outcomes can emerge and persist over time. The novel perspective of our model lies in its portrayal of how the firm's initial evaluation abilities, tied to historical disadvantages and beliefs about groups, can create self-reinforcing cycles that result in long-term disparities. Notably, two groups with identical skill distributions might differ significantly in hiring outcomes because of these initial perceptions.

We believe that this paper serves as a bridge between the literatures of statistical discrimination and algorithmic fairness, as well as contributing to each of these domains. On the statistical discrimination front, our work provides a dynamic generalization of a classical model, where the dynamic nature provides insight on why and how disparities evolve over time. On the algorithmic fairness front, our model provides a *framework* that models how the performance of machine learning algorithms change over time based on the availability of data, which depend on past decisions. This framework can be leveraged and extended to settings beyond the hiring application considered in this paper.

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Appendix A: Proofs for Section 3

This appendix includes the omitted proofs from Section 3.

Proof of Lemma 1

Let us define the following functions:

$$B(x) = \tau\sqrt{1+x} \quad \text{and} \quad f(x) = \frac{1}{P^b(1-\Phi(B(x)))^b}.$$

We would like to show that $f(x)$ intersects the line $y = x$ at most twice for $x \geq 0$. We have that $f(x) = x$ if and only if $x(1-\Phi(B(x)))^b = 1/P^b$. We define $v(x) = x(1-\Phi(B(x)))^b$ and prove that $v(x)$ intersects the horizontal line $y = 1/P^b$ at most twice. To prove this, we can show that $v'(x) = 0$ at most once. We have

$$v'(x) = (1-\Phi(B(x)))^b - xb(1-\Phi(B(x)))^{b-1}\Phi'(B(x))B'(x). \tag{3}$$

Then, $v'(x) = 0$ if and only if

$$\frac{1-\Phi(B(x))}{\Phi'(B(x))} = bxB'(x). \tag{4}$$

Let $L(x) = \frac{1-\Phi(B(x))}{\Phi'(B(x))}$ and $R(x) = bxB'(x)$ be the LHS and the RHS of (4) respectively. If we show that $L(x)$ is strictly decreasing in x and $R(x)$ is strictly increasing in x, then $L(x)$ would intersect $R(x)$ at most once, which completes the proof.

Note that if $h(x) = \frac{1-\Phi(x)}{\Phi'(x)}$, we have that $h'(x) < 0$ because

$$\begin{aligned} h'(x) &= \frac{-\Phi'(x)\Phi'(x) - (1-\Phi(x))\Phi''(x)}{(\Phi'(x))^2} \\ &= -1 + \frac{x(1-\Phi(x))}{\Phi'(x)} \\ &< 0, \end{aligned}$$

where the second equality follows from using $\Phi''(x) = -x\Phi'(x)$, and the inequality follows from using $1 - \Phi(x) < \Phi'(x)/x$.[4] Then, since $B(x) = \tau\sqrt{1+x}$ is strictly increasing and $h(x)$ is strictly decreasing, $L(x) = h(B(x))$ is strictly decreasing.

Next, consider the RHS of Eq. (4). We have

$$B'(x) = \frac{1}{2}\tau(1+x)^{-1/2}.$$

[4] To see this inequality, notice that, for any $t \in \mathbb{R}_+$, since $x/t > 1$ for x in (t, ∞), we have

$$\begin{aligned} 1-\Phi(t) &= \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-x^2/2}\,dx \\ &< \frac{1}{\sqrt{2\pi}}\int_t^\infty \frac{x}{t}e^{-x^2/2}\,dx \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{t}e^{-t^2/2} = \frac{\Phi'(t)}{t}. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} R'(x) &= \frac{b}{2}\tau(1+x)^{-1/2} - \frac{b}{4}x\tau(1+x)^{-3/2} \\ &= \frac{b}{2}\tau(1+x)^{-1/2}\left(1-\frac{1}{2}\cdot\frac{x}{1+x}\right) \\ &> 0. \end{aligned}$$

This establishes that $L(x)$ is strictly decreasing and $R(x)$ is strictly increasing. Hence, they intersect at most once. ■

Proof of Theorem 1

We keep using the following shorthand notation:

$$f(x) = \frac{1}{P^b(1-\Phi(\tau\sqrt{1+x}))^b}.$$

Notice that $f(0) = \frac{1}{P^b(1-\Phi(\tau))^b} > 0$.

We will show that there exists a $\bar{\tau}$ such that if $\tau > \bar{\tau}$, $f(x)$ is strictly above the $y = x$ line for all $x \geq 0$, while if $\tau < \bar{\tau}$, then $f(x)$ has two intersections with $y = x$, where $f(x)$ is below $y = x$ only in between the two intersections. Note that if $f(x)$ is always above $y = x$, then no matter what the initial noise variance is, the variance will keep increasing at each time step, and will approach ∞ as $t \to \infty$. Therefore, if $f(x)$ is always above the $y = x$ line, then there is only one steady state which is inactive. In the other case where $f(x)$ has two intersections with $y = x$, let $\bar{\sigma}^2$ is the second intersection of these curves. Then, any $\sigma^2 < \bar{\sigma}^2$ converges to the first intersection of the curves, which is an active stable steady state. Any $\sigma^2 > \bar{\sigma}^2$ converges to the inactive steady state. The second intersection, $\bar{\sigma}^2$, is also an active steady state, but it is not stable.

Therefore, to prove Theorem 1, we simply have to show the existence of the threshold $\bar{\tau}$ where $f(x)$ is above $y = x$ for $\tau > \bar{\tau}$, and $f(x)$ has two intersections with $y = x$ for $\tau < \bar{\tau}$.

First, we show that for any $\tau > \sqrt{2P^b/b}$, $f(x)$ is above the $y = x$ line for all $x \geq 0$. Since $f(0) > 0$, we will simply show that $f'(x) > 1$ for all $x \geq 0$.

$$\begin{aligned} f'(x) &= \frac{b\Phi'(\tau\sqrt{1+x})\frac{\tau}{2}\frac{1}{\sqrt{1+x}}}{P^b(1-\Phi(\tau\sqrt{1+x}))^{b+1}} \\ &\overset{(a)}{\geq} \frac{b\left(1-\Phi(\tau\sqrt{1+x})\right)\frac{\tau^2}{2}}{P^b(1-\Phi(\tau\sqrt{1+x}))^{b+1}} \\ &= \frac{b\tau^2}{2P^b}\frac{1}{(1-\Phi(\tau\sqrt{1+x}))^b} \overset{(b)}{>} 1, \end{aligned}$$

where (a) follows from using $1 - \Phi(x) < \Phi'(x)/x$ and (b) follows from $\tau^2 > 2P^b/b$.

Next, we will show that when τ is small enough, $f(x)$ has exactly two intersections with $y = x$, where $f(x)$ is below the line $y = x$ only in between the two intersections. Note that for any τ, if x is large enough, then $f'(x) > 1$. This can be seen using the previous set of inequalities:

$$f'(x) \geq \frac{b\tau^2}{2P^b}\frac{1}{\left(1-\Phi(\tau\sqrt{1+x})\right)^b} \overset{(a)}{>} 1,$$

where (a) follows by noting that for large enough x, $1-\Phi(\tau\sqrt{1+x})$ becomes close to 0 and therefore for $\tau < \sqrt{2P^b/b}$, we have $(1-\Phi(\tau\sqrt{1+x}))^b < \frac{b\tau^2}{2P^b}$.

Since $f(0)>0$ for all τ, we will show that when τ is small enough, there exists a $x_0>0$ where $f(x_0)<x_0$. Let $x_0 = \left(\frac{1}{\tau}\right)^2 - 1$. Then, $f(x_0) = \frac{1}{P^b(1-\Phi(1))^b}$, and hence $f(x_0)<x_0$ if $\tau < 1/\sqrt{(P^b(1-\Phi(1))^b)^{-1}+1}$. Therefore, if $\tau < \min\{1, 1/\sqrt{(P^b(1-\Phi(1))^b)^{-1}+1}\}$, then there exists a $x_0>0$ such that $f(x_0)<x_0$. Hence, for τ small enough, $f(x)$ has two intersections with $y=x$.

We have shown that if τ is large enough, $f(x)$ has no intersections with $y=x$, while if τ is small enough, it has two intersections. Note that taking x' as fixed, the value of $f(x')$ strictly and monotonically increases as τ increases. Therefore, let $\bar{\tau}$ be the smallest value of τ where $f(x)$ has fewer than 2 intersections with $y=x$. Then, it must be that for $\tau \geq \bar{\tau}$, $f(x)$ has at most one intersection with $y=x$, in which there is only one inactive steady state. For $\tau<\bar{\tau}$, $f(x)$ has two intersections with $y=x$. This completes the proof. ■

Proof of Proposition 1

Note that by using Lemma 1 and Theorem 1, $\bar{\sigma}(P)$ is the second intersection of $f(x)$ with $y=x$. By increasing P, the function

$$f(x,P) = \frac{1}{P^b\left(1-\Phi(\tau\sqrt{1+x})\right)^b}$$

decreases for all $x \in \mathbb{R}_+$. Therefore, the second intersection of $f(x)$ with $y=x$ increases as P increases, proving the first statement of the proposition.

For the second statement of the proposition, suppose there are two groups with populations P_1 and P_2 where $P_1<P_2$. We will show that $q(\tau,\sigma_*^2(P_1,\tau)) \leq q(\tau,\sigma_*^2(P_2,\tau))$. First, suppose group 1 has no active state; i.e., $\sigma_*^2(P_1,\tau)=\infty$. Then, $q(\tau,\sigma_*^2(P_1,\tau))=0$ and hence the inequality holds. Second, suppose $\sigma_*^2(P_1,\tau)<\infty$ is finite. Suppose that x_1^* (correspondingly, x_2^*) is the noise variance in the active steady state when the population size is P_1 (correspondingly, P_2). We also let q_1^* and q_2^* be the hiring probability in the active steady state for the two groups, respectively. Then, by definition, $f(x_1^*,P_1) = 1/(P_1q_1^*)^b$ and $f(x_2^*,P_2)=1/(P_2q_2^*)^b$. Note that $f(x,P_1) = \frac{P_2^b}{P_1^b} f(x,P_2)$ for any $x \geq 0$. Since $f(x,P_1) > f(x,P_2)$, and x_g^* corresponds to the first intersection of $f(x,P_g)$ with $y=x$, we have that $x_1^*>x_2^*$. Then,

$$1/(P_1q_1^*)^b = f(x_1^*,P_1) = \frac{P_2^b}{P_1^b} f(x_1^*,P_2) > \frac{P_2^b}{P_1^b} f(x_2^*,P_2), = \frac{P_2^b}{P_1^b} 1/(P_2q_2^*)^b \tag{5}$$

where the inequality follows from $x_1^*>x_2^*$ and that f is increasing in x. Rearranging the first and last terms results in $q_2^*>q_1^*$ as desired. ■

Proof of Proposition 2

Let $q = q(\tau;\sigma_*^2(P,\tau))$ be the proportion of applicants hired at the active steady state. Then, if an intervention hires q proportion of applicants in a single round, then the noise variance at the next time step will be $\sigma_*^2(P,\tau))$. Therefore, under the natural hiring rule, the noise variance will stay at this value by definition of the steady state, proving this is a sufficient intervention. ■

Proof of Proposition 3

Let $\sigma_*^2(A)$ be the noise variance corresponding to the active state for group A. Suppose $P(B) \geq P(A)$. Then, if $q_t(B) = q_*(A)$, then it will be that $\sigma_{t+1}^2(B) \leq \sigma_*^2(A)$. Proposition 1 states that $\bar{\sigma}^2(B) \geq \bar{\sigma}^2(A)$. Therefore, we have that $\sigma_{t+1}^2(B) \leq \sigma_*^2(A) \leq \bar{\sigma}^2(A) \leq \bar{\sigma}^2(B)$. Since the noise variance at time $t+1$ is below the threshold $\bar{\sigma}^2(B)$, group B will converge to the active steady state.

Next, suppose $P(B) < P(A)$ and the active steady state for group B exists. Let $\sigma_*^2(B)$ be the noise variance corresponding to the active state for group B, and let $q_*(B) \in [0,1]$ be the corresponding hiring rate at the steady state. From Proposition 1, $\sigma_*^2(B) > \sigma_*^2(A)$, and therefore $q_*(B) < q_*(A)$. Since $q_t(B) = q_*(A) > q_*(B)$, it must be that $\sigma_{t+1}^2(B) < \sigma_*^2(B) < \bar{\sigma}^2(B)$. Therefore, group B will converge to the active steady state. ■

Appendix B: Proofs of Competition Results

B.1. Preliminary Terms and Results

For group g at time t, we let

$$\gamma_t(g) = \frac{1}{1 + \sigma_t^2(g)} \tag{6}$$

denote the variance of the distribution of $\mathbb{E}[S_t \mid O_t]$. That is, $\mathbb{E}[S_t \mid O_t] \sim N(0, \gamma)$.

The key idea of the proofs for the competition model revolves around analyzing a function $\tau_g(q)$. For a group g, we define the function $\tau_g(q)$ for $q \in [0,1]$ to be the threshold such that for a group with noise variance $\sigma^2 = 1/(P(g)q)^b$, the threshold $\tau_g(q)$ results in exactly q percent of applicants hired. That is, if $q_{t-1}(g)$ proportion of applicants from group g were hired at time $t-1$ and the threshold at time t is $\tau_g(q_{t-1}(g))$, then we have that $q_t(g) = q_{t-1}(g)$. The function $\tau_g(q)$ can be explicitly written as

$$\tau_g(q) = \frac{\Phi^{-1}(1-q)}{\sqrt{1 + 1/(Pq)^b}}. \tag{7}$$

If the threshold at time t, τ_t, is greater than $\tau_g(q_{t-1}(g))$, this implies that the proportion hired from group g at time t, $q_t(g)$, is less than $q_{t-1}(g)$. Conversely, if $\tau_t < \tau_g(q_{t-1})$, then we will have $q_t(g) > q_{t-1}(g)$. Therefore, whether the hiring rate increases or decreases for group g depends on whether the threshold τ_t is less than or greater than $\tau_g(q_{t-1}(g))$.

We first prove several properties of $\tau_g(q)$.

Proposition 9. *The function $\tau_g(q)$ satisfies the following properties:*
(i) $\tau_g(q) \to 0$ as $q \to 0$.
(ii) There exists a $q' > 0$ such that $\tau_g'(q) > 0$ for all $q \in (0, q')$.
(iii) There exists a $q' > 0$ such that $\tau_g'(q) < 0$ for all $q \in (q', 0.5)$.

Proof of Proposition 9 We first state a useful upper bound on the inverse CDF of the normal distribution that holds for all $q \in [0, 0.5]$:

$$\Phi^{-1}(1-q) \leq \sqrt{2\log(1/2q)}. \tag{8}$$

Proof of property (i). Denote by $f_1(q) = \Phi^{-1}(1-q)$ and $f_2(q) = \sqrt{1 + 1/(Pq)^b}$ the numerator and denominator of $\tau(q)$ respectively. Note that when $q \to 0$, both $f_1(q) \to \infty$ and $f_2(q) \to \infty$. We take the derivative of both terms and then use L'Hopital's rule.

We have $f_1'(q) = -\frac{1}{\phi(\Phi^{-1}(1-q))}$, where $\phi(x)$ is the pdf of the normal distribution, and $f_2'(q) = \frac{-b}{2P^b q^{b+1}\sqrt{1+1/(Pq)^b}}$. Therefore,

$$\begin{aligned}
\lim_{q\to 0}\frac{f_1(q)}{f_2(q)} = \lim_{q\to 0}\frac{f_1'(q)}{f_2'(q)} &= \lim_{q\to 0}\frac{2P^b q^{b+1}\sqrt{1+1/(Pq)^b}}{b\phi(\Phi^{-1}(1-q))} \\
&= \lim_{q\to 0}\frac{2P^b}{b} q^{b+1}\sqrt{1+1/(Pq)^b}\cdot\sqrt{2\pi}\exp((\Phi^{-1}(1-q))^2/2)
\end{aligned}$$

Using (8), we get

$$\begin{aligned}
\lim_{q\to 0}\frac{f_1(q)}{f_2(q)} &\le \lim_{q\to 0}\frac{2P^b}{b} q^{b+1}\sqrt{1+1/(Pq)^b}\cdot\frac{\sqrt{2\pi}}{2q} \\
&= \frac{P^b\sqrt{2\pi}}{b}\lim_{q\to 0}\frac{\sqrt{1+1/(Pq)^b}}{q^{-b}} \\
&= \frac{P^b\sqrt{2\pi}}{b}\lim_{q\to 0}\frac{-\frac{1}{2\sqrt{1+1/(Pq)^b}}b(Pq)^{-b-1}P}{-bq^{-b-1}} \\
&= 0,
\end{aligned}$$

where the second last step uses L'Hopital's rule.

Proof of property (ii). Recall that $\tau(q) = \frac{f_1(q)}{f_2(q)}$, where $f_1(q) = \Phi^{-1}(1-q)$ and $f_2(q) = \sqrt{1+1/(Pq)^b}$.

$$\begin{aligned}
\tau'(q) &= \frac{f_1'(q)}{f_2(q)} - \frac{f_1(q)f_2'(q)}{(f_2(q))^2} \\
&= \frac{1}{\sqrt{1+1/(Pq)^b}}\left(f_1'(q) + \frac{bf_1(q)}{2q(P^bq^b+1)}\right). \qquad (9)
\end{aligned}$$

We want to show that $f_1'(q) + \frac{bf_1(q)}{2q(P^bq^b+1)} > 0$ when q is close to 0. Recall that $f_1'(q) = -\frac{1}{\phi(\Phi^{-1}(1-q))}$. As $q\to 0$, $\frac{bf_1(q)}{2q(P^bq^b+1)} \to \infty$ and $f_1'(q) \to -\infty$. We will show that $\frac{bf_1(q)}{2q(P^bq^b+1)} \to \infty$ at a faster rate than $f_1'(q) \to -\infty$. Consider the fraction of these two terms:

$$\frac{\left(\frac{bf_1(q)}{2q(P^bq^b+1)}\right)}{-f_1'(q)} = \frac{b\Phi^{-1}(1-q)\phi(\Phi^{-1}(1-q))}{2q(P^bq^b+1)}.$$

Note that $\phi(x)$ is a decreasing function in $x > 0$. Using (8), we have $\phi(\Phi^{-1}(1-q)) \ge \phi(\sqrt{2\log(1/(2q))}) = 2q/\sqrt{2\pi}$. Therefore,

$$\frac{\left(\frac{bf_1(q)}{2q(P^bq^b+1)}\right)}{-f_1'(q)} \ge \frac{b\Phi^{-1}(1-q)}{\sqrt{2\pi}(P^bq^b+1)}.$$

Note that $\frac{b\Phi^{-1}(1-q)}{\sqrt{2\pi}(P^bq^b+1)} \to \infty$ as $q\to 0$, and $\tau'(q) > 0$ for any q that satisfies $\frac{b\Phi^{-1}(1-q)}{\sqrt{2\pi}(P^bq^b+1)} > 1$. Therefore, there exists a $q' > 0$ such that $\tau'(q) > 0$ for all $q \in (0, q')$.

Proof of property (iii). From the definition of $\tau'(q)$ from (9), it is easy to check that $\tau'(0.5) < 0$; note that $f_1'(0.5) < 0$ and $f_1(0.5) = 0$. Since $\tau'(q)$ is continuous, we have that there exists a q' such that $\tau'(q) < 0$ for all $q \in (q', 0.5]$. □

B.2. Preliminary Results on the Dynamics

We now prove results on the convergence of the system. The direction of movement of the hiring rates depend crucially on how the value of $\tau_A(q_t(A))$ compares with $\tau_B(q_t(B))$.

LEMMA 2. *If $\tau_A(q_t(A)) > \tau_B(q_t(B))$, then $q_{t+1}(A) > q_t(A)$. If $\tau_A(q_t(A)) < \tau_B(q_t(B))$, then $q_{t+1}(A) < q_t(A)$. If $\tau_A(q_t(A)) = \tau_B(q_t(B))$, then $q_{t+1}(A) = q_t(A)$.*

Next, we show that a steady state $(q_A^\infty, q_B^\infty) > 0$ must satisfy $\tau_A(q_A^\infty) = \tau_B(q_B^\infty)$.

LEMMA 3. *If $q_t(A) \to q_A^\infty$ and $q_t(B) \to q_B^\infty$ as $t \to \infty$ where $q_A^\infty > 0$ and $q_B^\infty > 0$, then it must be that $\tau_A(q_A^\infty) = \tau_B(q_B^\infty)$.*

Lastly, we show show that if $\tau_A(q_t(A)) > \tau_B(q_t(B))$, then $q_t(A)$ will converge to the closest point to the right where $\tau_A(q_t(A)) = \tau_B(q_t(B))$ is equal to zero. Similarly, if $\tau_A(q_t(A)) < \tau_B(q_t(B))$, $q_t(A)$ will converge to the closest point to the left where $\tau_A(q_t(A)) = \tau_B(q_t(B))$ is equal to zero.

LEMMA 4. *Suppose $\tau_A(q_t(A)) > \tau_B(q_t(B))$, and suppose $\tilde{q} = \inf\{q > q_t(A) : \tau_A(q_t(A)) = \tau_B(q_t(B))\}$ exists. Then, $\lim_{t\to\infty} q_t(A) = \tilde{q}$. Similarly, suppose $\tau_A(q_t(A)) < \tau_B(q_t(B))$, and suppose $\tilde{q} = \sup\{q < q_t(A) : \tau_A(q_t(A)) = \tau_B(q_t(B))\}$ exists. Then, $\lim_{t\to\infty} q_t(A) = \tilde{q}$.*

Proof of Lemma 2. Suppose $\tau_A(q_0(A)) < \tau_B(q_0(B))$. We claim that the threshold at time $t = 1$, τ_1 is strictly between $\tau_A(q_0(A))$ and $\tau_B(q_0(B))$. Suppose instead, $\tau \le \tau_A(q_0(A))$. Then it would be that $q_1(A) \ge q_0(A)$ and $q_1(B) > q_0(B)$. This is a contradiction that both groups increase the proportion hired. Similarly, it will be a contradiction if $\tau \ge \tau_A(q_0(B))$. Therefore, $\tau_1 \in (\tau_A(q_0(A)), \tau_B(q_0(B)))$.

Since $\tau_1 > \tau_A(q_0(A))$, it will be that $q_1(A) < q_0(A)$, and since $\tau_1 < \tau_B(q_0(A))$, it will be that $q_1(B) < q_0(B)$. Then, applying Proposition 4 gives us the result. $\square$

Proof of Lemma 3 Suppose, to the contrary, that $q_t(A) \to q_A^\infty$ and $q_t(B) \to q_B^\infty$ where $q_A^\infty > 0$ and $q_B^\infty > 0$, and $\tau_A(q_A^\infty) \neq \tau_B(q_B^\infty)$. Let $\tau_A = \tau(q_A^\infty)$ and $\tau_B = \tau(q_B^\infty)$. Suppose, WLOG, $\tau_A < \tau_B$. Let $\tau_1 = \tau_A + (\tau_B - \tau_A)/4$, $\tau_2 = \tau_A + (\tau_B - \tau_A)/2$ and $\tau_3 = \tau_A + 3(\tau_B - \tau_A)/4$ be three points evenly spaced between τ_A and τ_B. Let $Q_g(\tau, q) = 1 - \Phi(\tau\sqrt{1 + 1/(P(g)q)^b})$ denote the percent of applicants hired from group g when the previous proportion of applicants hired was q (and hence the noise in this round is $1/(P(g)q)^b$), and the threshold is τ.

Let $q'_A = Q_A(\tau_1, q_A^\infty)$ be the proportion hired if the previous proportion hired was q_A^∞ and the threshold if τ_1. Since $\tau_1 > \tau_A$, we have that $q'_A < q_A^\infty$. Since $q_t(A) \to q_A^\infty$, let t_0 be large enough so that $q_t(A) > q'_A$ for all $t \ge t_0$. That is, if $t \ge t_0$, then the percentage of applicants hired will be greater than q'_A henceforth.

Next, consider the threshold τ_2. We will show that it must be the case that the chosen threshold should be strictly smaller than τ_2. We have that $Q_A(\tau_2, q_A^\infty) < q'_A$. When τ_2 is fixed, $g(\tau_2, \cdot)$ is a continuous function in the second argument. Therefore, there exists an $\epsilon > 0$ such that if $|q - q_A^\infty| \le \epsilon$, then $Q_A(\tau_2, q) < q'_A$. Let t_1 be large enough such that for all $t \ge t_1$, we have that $|q_t(A) - q_A^\infty| \le \epsilon$. Therefore, when $t \ge t_1$, a threshold of τ_2 results in the proportion of applicants being hired to be strictly less than q'_A.

Combining the two above arguments, if $t \ge \max(t_0, t_1)$, since it must be that $q_t(A) > q'_A$, but a threshold of τ_2 results in $g(\tau_2, q_t(A)) < q'_A$, it must be that the threshold τ_t is *strictly less* than τ_2.

Now, one can repeat the same argument with the thresholds τ_3 and τ_2 for group B (where τ_3 is the analog of τ_1 in the above argument) to show that it must be that the threshold τ_t is *strictly higher* than τ_2. This results in a contradiction. □

Proof of Lemma 4. Suppose $\tau_A(q_t(A)) > \tau_B(q_t(B))$, and suppose $q_A = \inf\{q > q_A : D(q) = 0\}$ exists. Let $q_B = (C - P(A)q_A)/P(B)$ be the corresponding hiring rate for B. (q_A, q_B) is consistent and corresponds to a steady state. Let $\gamma_A = 1/(1 + 1/(P(A)q_A))$ and $\gamma_B = 1/(1 + 1/(P(B)q_B))$ be the corresponding γ at this steady state. From Lemma 2, we know that $q_{t+1}(A) > q_t(A)$.

First, we show that $q_{t+1}(A)$ is at most $\tilde{q}$. Let τ_t be the threshold used at time t. Suppose, to the contrary, that $q_{t+1}(A) > \tilde{q}$. Since $q_t(A) < q_A$, $\gamma_t(A) < \gamma_A$. Therefore, the threshold τ_t must be smaller than $\tau_A(q_A)$. Since $q_t(B) > q_B$, $\gamma_t(B) > \gamma_B$. Then, under the threshold $\tau_t < \tau_A(q_A) = \tau_B(q_B)$, the proportion hired for group B would be larger than q_B. That means $q_{t+1}(A) > q_A$ and $q_{t+1}(B) > q_B$, which is a contradiction since $P(A)q_A + P(B)q_B = C$.

Therefore we have shown that $q_t(A)$ increases in t, and $q_t(A) \leq \tilde{q}$ for all t. Then, it must be that $\lim_{t\to\infty} q_t(A) = \tilde{q}$ from Lemma 3. The second statement of the lemma can be shown in the same way as the first. □

B.3. Main Proofs

Proof of Proposition 4. For a fixed $t \geq 1$, suppose $q_{t-1}(A) < q_t(A)$. We will show that $q_t(A) < q_{t+1}(A)$. Since $P(A)q_s(A) + P(B)q_s(B) = C$ for any s, if the mass hired from group A increased $(q_{t-1}(A) < q_t(A))$, it must be that the mass hired from group B decreased; $q_{t-1}(B) > q_t(B)$. Therefore, $\gamma_{t+1}(A) > \gamma_t(A)$ and $\gamma_{t+1}(B) < \gamma_t(B)$.

Let τ' be the threshold such that $\Pr(\mathbb{E}[S_{t+1}(A) \mid O_{t+1}(A)] > \tau') = q_t(A)$. τ' is the threshold that yield the mass hired from group A to be the same from time t and $t+1$. We will show that $\tau_{t+1} < \tau'$, and hence the mass hired from group A will be larger at time $t+1$ compared to t. Since $\gamma_{t+1}(A) > \gamma_t(A)$, we have that $\tau' > \tau_t$. Since $\gamma_{t+1}(B) < \gamma_t(B)$, the mass hired from group B at time $t+1$ if the threshold was τ' is smaller than $q_t(B)$. Therefore, under τ', the total mass hired would be strictly less than C. Hence, it must be that $\tau_{t+1} < \tau'$. That means that $q_{t+1}(A)$, the mass hired from group A at time $t+1$, will be larger than $q_t(A)$. The same argument can be used to show that if $q_t(A) < q_{t-1}(A)$, then $q_{t+1}(A) < q_t(A)$.

Lastly, if $q_t(A) = q_{t-1}(A)$, then $q_t(B) = q_{t-1}(B)$. Therefore, since nothing changed from time $t-1$ to t, the threshold $\tau_t = \tau_{t-1}$, hence we will have $q_{t+1}(A) = q_t(A)$.

The limit $\lim_{t\to\infty} q_t(A)$ exists due to the monotone convergence theorem. □

Proof of Theorem 2. We assume $g = A$ and $g' = B$, but the same proof works if $g = B$ and $g' = A$. Let $q_B = C/P(B)$ be the proportion hired from group B if no one from group A is hired. Let $\delta > 0$ such that if $|q - q_B| \leq \delta$, then $\tau_B(q) > \tau_B(q_B)/2$. That is, if $q_t(B)$ is within δ of q_B, then $\tau_B(q_t(B))$ is lower bounded by $\tau_B(q_B)/2$. Let $q' \in (0, \delta P(B)/P(A))$ be small enough that $\tau_A(q) < \tau_B(q_B)/2$ for all $q \in (0, q')$ (this is possible since $\tau_A(q) \to 0$ as $q \to 0$ from Proposition 9.

Then, suppose $q_0(A) < q'$. Since $q_0(A) < \delta P(B)/P(A)$, it must be that $q_0(B) \geq q_B - \delta$. Therefore, $\tau_B(q_0(B)) > \tau_B(q_B)/2$. Since $q_0(A) < q'$, $\tau_A(q_0(A)) < \tau_B(q_B)/2 < \tau_B(q_0(B))$.

Therefore, $q_1(A) < q_0(A)$ and $q_1(B) > q_1(B)$. By Proposition 4, $q_t(A)$ decreases over time and converges to some limit q_A^∞, and $q_t(B)$ increases over time and converges to some limit q_B^∞. Moreover, we have that $q_t(A) < \tau_B(q_B)/2$ and $q_t(B) > \tau_B(q_B)/2$ for all $t > 0$. Therefore, using Lemma 3, it cannot be that $\tau_A(q_A^\infty) = \tau_B(q_B^\infty)$. Hence, it must be that $q_A^\infty = 0$. □

Proof of Theorem 3. Let $q' > 0$ such that $\tau_B(q)$ is strictly increasing in q for all $q \leq q'$ (such a q' exists due to Proposition 9).

Let $C' = q' P(B)$, and let $C < C'$. We will show that $\tau_A(q_0(A)) < \tau_B(q_0(B))$. Note that since $P(A) \leq P(B)$, we have $\tau_A(q) \leq \tau_B(q)$ for any q. The inequality is strict if $P(A) < P(B)$.

Hence if $q_0(A) = q_0(B)$ and $P(A) < P(B)$, then we have $\tau_A(q_0(A)) < \tau_B(q_0(A)) = \tau_B(q_0(B))$. Next, if $q_0(A) < q_0(B)$, we have $\tau_A(q_0(A)) \leq \tau_B(q_0(A)) < \tau_B(q_0(B))$, where the second inequality follows from the fact that τ_B is increasing for $q \in q'$.

Hence we have $\tau_A(q_0(A)) < \tau_B(q_0(B))$ Then, by Lemma 2, $q_t(A)$ decreases over time and converges to some limit q_A^∞, and $q_t(B)$ increases over time and converges to some limit q_B^∞, where $q_B^\infty > q_A^\infty$. Then it must be that $q_A^\infty = 0$. If it was not, then by Lemma 3, it must be that $\tau_A(q_A^\infty) = \tau_B(q_B^\infty)$, which is not possible since $\tau_B(\cdot)$ is strictly increasing under the domain of interest, and $\tau_A(q) \leq \tau_B(q)$ for all q. □

Proof of Theorem 4. Let $q' \in (0.0.5)$ such that $\tau_A(q)$ is strictly decreasing in q for all $q \in [q', 0.5)$; such a q' exists from property (iii) of Proposition 9. Let $C' = 2q'$. Fix any $C \in (C', 1)$. The corresponding steady state is the one where $q_A = q_B = C/2$. We would like to show that if the initial hiring rates start close to $q_A = q_B = C/2$, then we still converge to this steady state.

Let $\delta > 0$ such that $\tau_A(q)$ is decreasing for all $q \in (C/2 - \delta, C/2 + \delta)$. Suppose $q_0(A) < q_0(B)$, and that $q_0(A) > C/2 - \delta$. Then, $\tau_A(q_0(A)) > \tau_B(q_0(B))$. By Lemma 4, we have that $q_t(A)$ converges to $\inf\{q > q_0(A) : \tau_A(q_t(A) = \tau_B(q_t(B))\} = C/2$. Therefore, if $q_0(A)$ and $q_0(B)$ start close to $C/2$, then we converge to $q_\infty(A) = q_\infty(B) = C/2$. □

Proof of Theorem 5. Let $q' \in (0.0.5)$ such that $\tau_A(q)$ and $\tau_B(q)$ are both strictly decreasing in q for all $q \in (q', 0.5)$; such a q' exists from property (iii) of Proposition 9. Note that $\tau_A(q') < \tau_B(q')$ if $P(A) < P(B)$.

Let $\bar{\tau} = \tau_A(q')$. Let $\tau_g^{-1}(\tau)$ be the inverse of $\tau_g(q)$ over $q \in (q', 0.5)$, which exists since $\tau_g(q)$ is strictly decreasing over this interval. Then, any $\tau \in (0, \bar{\tau})$ corresponds to a steady state for A and B for the capacity corresponding to $C = P(A)\tau_A^{-1}(\tau) + P(B)\tau_B^{-1}(\tau)$. Moreover, as τ goes from $\bar{\tau}$ down to 0, the corresponding capacity C increases from some value C' to $0.5(P(A) + P(B))$. (C' is the capacity corresponding to the steady state for $\bar{\tau}$.)

Therefore, for any $C \in (C', 0.5(P(A) + P(B))$, there exists a $q_A, q_B \in (q', 0.5)$ such that $\tau_A(q_A) = \tau_B(q_B)$ and $P(A)q_A + P(B)q_B = C$. This shows the first part of the theorem.

Let $\delta > 0$ such that for all $q \in (q_A - \delta, q_A)$, $\tau_A(q) > \tau_A(q_A)$. We claim that under capacity C, if $q_0(A) \in (q_A - \delta, q_A)$, then we converge to the outcome (q_A, q_B). By construction, we have $\tau_A(q_0(A)) > \tau_A(q_A) = \tau_B(q_B) < \tau_B(q_0(B))$. Therefore, by Lemma 2, $q_t(A)$ will increase over time and $q_t(B)$ will decrease over time. We need to show that $q_t(A)$ must converge to q_A.

To prove this, we show that $q_t(A) < q_A$ and $q_t(B) > q_B$ for all t. We show this by induction. This is satisfied for $t = 0$; assume it holds for t. Suppose, to the contrary, that $q_{t+1}(A) > q_A$. Let $\gamma_A = 1/(1 + 1/(P(A)q_A))$

and $\gamma_B = 1/(1+1/(P(B)q_B))$ be the corresponding γ at the steady state. Since $q_t(A) < q_A$, $\gamma_t(A) < \gamma_A$. Therefore, the threshold τ_t must be smaller than $\tau_A(q_A)$. Since $q_t(B) > q_B$, $\gamma_t(B) > \gamma_B$. Then, under the threshold $\tau_t < \tau_A(q_A) = \tau_B(q_B)$, the proportion hired for group B would be larger than q_B. That means $q_{t+1}(A) > q_A$ and $q_{t+1}(B) > q_B$, which is a contradiction since $P(A)q_A + P(B)q_B = C$.

Lemma 3 states that $q_t(A)$ and $q_t(B)$ must converge to a limit wherein which $\tau_A(q) = \tau_B(q)$, and therefore they must converge to q_A and q_B.

□

B.4. Proof of Proposition 5

We will show

$$\frac{q_s^{(1)}(A)}{q_s^{(1)}(B)} < \frac{q_s^{(2)}(A)}{q_s^{(2)}(B)}. \tag{10}$$

This implies

$$\frac{P_s^{(1)}(A)q_s^{(1)}(A)}{P_s^{(1)}(B)q_s^{(1)}(B)} < \frac{P_s^{(2)}(A)q_s^{(2)}(A)}{P_s^{(2)}(B)q_s^{(2)}(B)},$$

since the ratio of the group sizes are the same across settings (1) and (2). Then,

$$\zeta_s^{(1)}(A) = \frac{P_s^{(1)}(A)q_s^{(1)}(A)}{P_s^{(1)}(A)q_s^{(1)}(A) + P_s^{(1)}(B)q_s^{(1)}(B)} < \frac{P_s^{(2)}(A)q_s^{(2)}(A)}{P_s^{(2)}(A)q_s^{(2)}(A) + P_s^{(2)}(B)q_s^{(1)}(B)} = \zeta_s^{(2)}(A),$$

completing the proof.

Our goal is to show (10). The hiring rate at time s for group g is the probability that the inferred skill $\mathbb{E}[S_s(g)|O_s(g)]$ is above the threshold τ. Note that the inferred skill has a distribution of $N(0, \gamma_s(g))$, where the $\gamma_s(g)$ is the same across settings (1) and (2). Since $\sigma_s(A) < \sigma_s^2(B)$, we have $\gamma_s(A) > \gamma_s(B)$ (as defined in (6)).

The difference between the two settings is the threshold τ. Let $\tau^{(1)}, \tau^{(2)}$ be the induced thresholds at time s for settings (1) and (2) respectively. Since the thresholds are defined to satisfy

$$P_s(A)\Pr(\mathbb{E}[S_s(A) \mid O_s(A)] > \tau) + P_s(B)\Pr(\mathbb{E}[S_s(B) \mid O_s(B)] > \tau) = C,$$

and we have that $P_s(A)$ and $P_s(B)$ are higher under setting (2), it must be that $\tau^{(2)} > \tau^{(1)}$. Then, to show (10), we prove the following lemma.

Lemma 5 **(Monotonicity of the normal–tail ratio)**. *Let $X \sim \mathcal{N}(0, \sigma_X^2)$, $Y \sim \mathcal{N}(0, \sigma_Y^2)$ with $0 < \sigma_Y < \sigma_X$. Define the function*

$$g(t) = \frac{\Pr(X > t)}{\Pr(Y > t)}, \qquad t > 0.$$

Then $g(t)$ is strictly increasing for $t > 0$.

Applying Lemma 5 implies that $g(\tau^{(1)}) < g(\tau^{(2)})$, which finishes the proof.

B.4.1. Proof of Lemma 5. We write the standard normal survival function and density as $\bar{\Phi}(z) = 1-\Phi(z)$ and $\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$. Put $a = \sigma_X^{-1}$, $b = \sigma_Y^{-1}$; thus $0 < a < b$. For every $t > 0$

$$g(t) = \frac{\bar{\Phi}(at)}{\bar{\Phi}(bt)}.$$

Set $f(t) = \log g(t)$. With the inverse Mills ratio $h(z) = \varphi(z)/\bar{\Phi}(z)$ $(z > 0)$,

$$f'(t) \;=\; -a\,h(at) + b\,h(bt).$$

Define $m(z) = z\,h(z) = z\varphi(z)/\bar{\Phi}(z)$. Differentiating,

$$m'(z) = \frac{\varphi(z)}{\bar{\Phi}(z)^2}\Big[(1-z^2)\bar{\Phi}(z) + z\varphi(z)\Big].$$

For $z > 0$ the classical Mills bound $\varphi(z) > z\bar{\Phi}(z)$ implies the bracketed term is positive, so $m'(z) > 0$. Hence m is strictly increasing on $(0,\infty)$.

Because $a < b$ and m is increasing, $m(bt) > m(at)$ for every $t > 0$. Dividing by $t > 0$ gives $b\,h(bt) > a\,h(at)$. Therefore, $f'(t) > 0$, and hence $g(t) = e^{f(t)}$ is strictly increasing for $t > 0$.

Appendix C: Proofs for Section 6.1

This section provides proofs for the results regarding the effort models.

LEMMA 6. *The optimal effort, $\eta^*(x)$ satisfies*

$$\eta^*(x) = \frac{1}{a}\Phi'((\tau - \eta^*(x))\sqrt{1+x})\sqrt{1+x}.$$

The proof of this lemma is immediate by taking the first order conditions of the definition of $\eta^*(x)$:

$$\eta^*(x) = \arg\max_{\eta \geq 0} 1 - \Phi((\tau-\eta)\sqrt{1+x}) - \frac{a}{2}\eta^2.$$

Next, we provide an upper bound on the optimal effort, $\eta^*(\sigma^2)$, as a function of τ.

LEMMA 7. $\eta^*(\sigma^2) \leq \alpha\tau$ *for a constant* $\alpha < 0.15$.

Proof of Lemma 7

Fix σ^2. If σ^2 is the noise variance and $\eta^*(\sigma^2)$ is the effort, then the distribution of $\mathbb{E}[S \mid O]$ is $\mathcal{N}(\eta^*(\sigma^2), \frac{1}{1+\sigma^2})$. Our goal is to show that $\eta^*(\sigma^2) \leq \alpha\tau$, where $\alpha < 0.15$ will be a constant defined later.

We provide a geometric interpretation of the optimal effort $\eta^*(\sigma^2)$, which we summarize in Fig. 8. Let $h(x) = \sqrt{\frac{1+\sigma_t^2}{2\pi}}\exp(-\frac{1}{2}(1+\sigma_t^2)x^2)$ be the pdf of $\mathcal{N}(0, \frac{1}{1+\sigma_t^2})$. Let $\ell(x) = -ax + a\tau$ be the line that represents the derivative of the effort cost. Let $A_\ell(z_1, z_2)$ and $A_h(z_1, z_2)$ denote the area under ℓ and h respectively between $x = z_1$ and $x = z_2$. Then, for $\eta > 0$, $A_\ell(\tau - \eta, \tau)$ represents the cost of putting in effort η, while $A_h(\tau-\eta, \tau)$ represents the increase in utility in putting in effort η, compared to putting in 0 effort. Therefore, the optimal effort η^* is the point that maximizes $A_h(\tau-\eta,\tau) - A_\ell(\tau-\eta,\tau)$.

Since η^* maximizes the difference in area under the curves of h and ℓ, it must occur at an intersection of those curves. Note that ℓ and h can have up to three intersections; this is denoted in Fig. 8 as x_1, x_2 and x_3. We show that the optimal effort occurs at the *last* intersection of ℓ and h.

CLAIM 1. $\eta^*(\sigma^2) = \tau - x_3$, *where x_3 is the largest intersection of ℓ and h.*

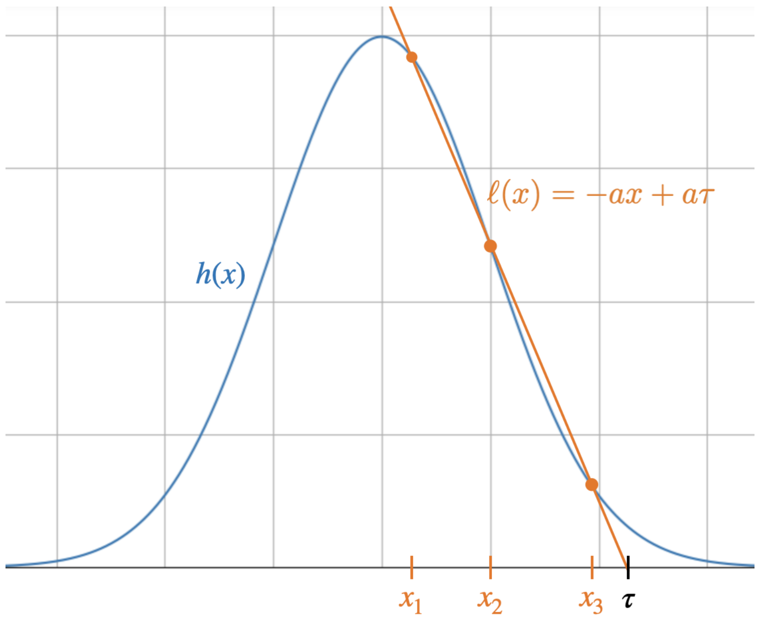

Figure 8 $h(x)$ **represents the distribution of** $\mathbb{E}[S \mid O]$**, and** $\ell(x)$ **represents the derivative of the effort cost. This plot shows an example of a case where the two curves intersect three times.**

We defer the proof of this claim to after the proof of this result.

Fix τ and a, and let $x_0 = R\tau$, where $R = \frac{1}{2}\left(1 + \sqrt{1 - 2\frac{\exp(-1/2)}{\sqrt{2\pi}}}\right) \approx 0.859$. We will show that $\ell(x_0) > h(x_0)$. If this is the case, then the last intersection of ℓ and h will be after x_0. Since $a \geq 2/\tau^2$, $\ell(x_0) = \ell(R\tau) \geq \frac{2(1-R)}{\tau}$. We will show that $h(R\tau) \leq \frac{2(1-R)}{\tau}$. More specifically, we will show this holds under *any* zero-mean normal distribution that h can represent as well as any τ. That is, we will show the following statement: for any $x > 0$, $\phi(x, \sigma) \leq \frac{2R(1-R)}{x}$ for *any* $\sigma > 0$, where $\phi(x, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{1}{2}\frac{x^2}{\sigma^2})$ is the pdf of the normal distribution with mean 0 and variance σ^2.

Fix $x > 0$. We will find the σ that results in the highest value of $\phi(x, \sigma)$. We take the derivative of $\phi(x, \sigma)$ with respect to σ:

$$\begin{aligned}
\phi_\sigma(x, \sigma) &= \frac{1}{\sqrt{2\pi}}\left(-\frac{1}{\sigma^2}\exp(-\frac{1}{2}\frac{x^2}{\sigma^2}) + \frac{1}{\sigma}\exp(-\frac{1}{2}\frac{x^2}{\sigma^2})(-\frac{1}{2}x^2)(-2)\sigma^{-3}\right) \\
&= \frac{\exp(-\frac{1}{2}\frac{x^2}{\sigma^2})}{\sigma^2\sqrt{2\pi}}\left(-1 + \frac{x^2}{\sigma^2}\right).
\end{aligned}$$

Then, $\phi_\sigma(x, \sigma) = 0$ when $x = \sigma$. Therefore, $\phi(x, \sigma)$ is maximized when $\sigma = x$. This yields the desired result: for any x and σ,

$$\begin{aligned}
\phi(x, \sigma) &= \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right) \\
&\leq \frac{1}{x\sqrt{2\pi}}\exp\left(-\frac{1}{2}\right) \\
&= \frac{2R(1-R)}{x}.
\end{aligned}$$

R was defined exactly so that the last step goes through. ■

Proof of Claim 1

Let $A_\ell(z_1, z_2)$ and $A_h(z_1, z_2)$ denote the area under ℓ and h respectively between $x = z_1$ and $x = z_2$. Recall that η^* is the point that maximizes $A_h(\tau - \eta^*, \tau) - A_\ell(\tau - \eta^*, \tau)$.

If ℓ and h have only one intersection, then it is clear that $\eta^* = \tau - x_0$, where x_0 is the intersection of the curves.

Now, assume that ℓ and h have three intersections as in Fig. 8. Our goal is to show that $\eta^* = \tau - x_3$, rather than $\eta^* = \tau - x_1$. We can see that clearly, if $\eta = \tau - x_3$, then the area under h is larger than the area under ℓ, and therefore the utility of putting in effort $\tau - x_3$ is larger than the utility of putting in zero effort. We will show that putting in effort $\eta = \tau - x_1$ (the first intersection) will result in a utility that is lower than putting in zero effort.

We will let $A_\ell(z_{1,2})$ and $A_h(z_1, z_2)$ denote the area under ℓ and h respectively between $x = z_1$ and $x = z_2$. Assume, to the contrary, that there exists three distinct points of contact, and that $A_h(x_1, \tau) \geq A_\ell(x_1, \tau)$. We will derive a contradiction. Recall that we assume $a \geq 2/\tau^2$, which implies that $A_\ell(0, \tau) \geq 1$.

Let σ^2 be the variance of the normal distribution that h represents. Note that the concavity of h changes from negative to positive as x increases from 0. It is easy to show that the inflection point of h occurs at $x = \sigma$, and the derivative of h at that point is $h'(\sigma) = -\frac{e^{-1/2}}{\sigma^2\sqrt{2\pi}}$. Then, if $\ell(x)$ has three points of contact with h, it must be that x_1 occurs during the parts when h is concave; that is, it must be that

$$x_1 < \sigma. \tag{11}$$

Moreover, the slope of the line $\ell(x)$ must be less steep than $h'(\sigma)$; that is, it must be that

$$a < \frac{e^{-1/2}}{\sigma^2\sqrt{2\pi}}.$$

If it was the case that $a \geq \frac{e^{-1/2}}{\sigma^2\sqrt{2\pi}}$, then the line ℓ would intersect h at most once.

Therefore, we have that a satisfies both $a \geq 2/\tau^2$, and $a < \frac{e^{-1/2}}{\sigma^2\sqrt{2\pi}}$. Combining this implies

$$\tau > \sigma\sqrt{2\sqrt{2\pi}e^{1/2}} > 2.87\sigma. \tag{12}$$

Obviously, $A_h(x_1, \tau) \leq 1/2$. But note that by assumption, we have $A_\ell(0, \tau) \geq 1$. Since we're assuming $A_h(x_1, \tau) \geq A_\ell(x_1, \tau)$, it must be that x_1 is large enough such that $A_\ell(x_1, \tau) \leq 1/2$. Therefore, it must be that

$$x_1 \geq (1 - 1/\sqrt{2})\tau.$$

Combining this with (12) results in

$$x_1 \geq (1 - 1/\sqrt{2})2.87\sigma > 0.84\sigma.$$

Now, we will go through the above logic one more time. Since $x_1 > 0.84\sigma$, we have $A_h(x_1, \tau) < 1 - \Phi(0.84) < 0.21$. Therefore, it must be that $A_\ell(x_1, \tau) < 0.21$. Using that $A_\ell(0, \tau) \geq 1$, it must be that $x_1 > (1 - \sqrt{0.21})\tau > 0.54\tau$. Using (12), we get

$$x_1 > 0.54\tau \geq 0.54(2.87\sigma) > 1.54\sigma.$$

This is a contradiction to (11), and we are done. ■

Proof of Theorem 6

We prove this theorem by using the following claims.

Claim 1: Similar to Lemma 1, for large enough a, the function

$$f(x) = \frac{1}{P^b \left(1 - \Phi(\left(\tau - \eta^*(x)\right) \sqrt{1+x}\right)^b}$$

where the optimal effort is given by

$$\eta^*(x) = \arg\max_{\eta \geq 0} 1 - \Phi((\tau - \eta)\sqrt{1+x}) - \frac{a}{2}\eta^2$$

intersects $y = x$ in at most two points.

Proof of Claim 1: Using the same notation and line of argument as the one in the proof of Lemma 1, we argue that

$$L(x) = \frac{1 - \Phi(B(x))}{\Phi'(B(x))},$$

where

$$B(x) = (\tau - \eta^*(x))\sqrt{1+x}$$

is decreasing in x and

$$R(x) = bxB'(x)$$

is increasing in x. Let us start by proving $L(x)$ is decreasing. We can write

$$L'(x) = \left(-1 + \frac{B(x)(1 - \Phi(B(x)))}{\Phi'(B(x))}\right) B'(x).$$

Notice that the first term is negative (because $1 - \Phi(t) < \Phi'(t)/t$ for all $t \in \mathbb{R}_+$), and therefore, it suffices to prove that $B(x)$ is increasing in x, which we prove next.

$B'(x)$ can be written as

$$B'(x) = -\eta_x^*(x)\sqrt{1+x} + \frac{1}{2}\frac{\tau - \eta^*(x)}{\sqrt{1+x}}.$$

Then to show that $B'(x) > 0$ is equivalent to showing $\eta_x^*(x) < \frac{\tau - \eta^*(x)}{2(1+x)}$. We now replace the term $\eta_x^*(x)$ with an expression that depends on $\eta^*(x)$. Recall that the first-order condition (Lemma 6) gives us

$$\eta^*(x) = \frac{1}{a}\Phi'((\tau - \eta^*(x))\sqrt{1+x})\sqrt{1+x}.$$

We take the derivative of both sides of the above equation, which yields

$$\begin{aligned} a\eta_x^*(x) = &\Phi''\left((\tau - \eta^*(x))\sqrt{1+x}\right)\left(-\eta_x^*(x)\sqrt{1+x} + (\tau - \eta^*(x))\frac{1}{2\sqrt{1+x}}\right)\sqrt{1+x} \\ &+ \Phi'((\tau - \eta^*(x))\sqrt{1+x})\frac{1}{2\sqrt{1+x}}. \end{aligned}$$

Simplifying the above expression and using $\Phi^{''}(y) = -y\Phi'(y)$ results in

$$\begin{aligned} \eta_x^*(x) = &\frac{1}{2}\frac{-\left(\tau - \eta^*(x)\right)^2\sqrt{1+x} + (1+x)^{-1/2}}{\frac{a}{\Phi'((\tau - \eta^*(x))\sqrt{1+x})} - (\tau - \eta^*(x))(1+x)^{3/2}} \\ = &\frac{\eta^*(x)\left(\frac{-1}{2}\left(\tau - \eta^*(x)\right)^2 + \frac{1}{2(1+x)}\right)}{1 - \eta^*(x)(\tau - \eta^*(x))(1+x)}. \end{aligned} \tag{13}$$

Using this, to show that $B'(x)>0$ is equivalent to showing

$$\frac{\eta^*(x)\left(\frac{-1}{2}\left(\tau-\eta^*(x)\right)^2+\frac{1}{2(1+x)}\right)}{1-\eta^*(x)(\tau-\eta^*(x))(1+x)}<\frac{\tau-\eta^*(x)}{2(1+x)}. \tag{14}$$

If the denominator of the LHS is positive, then (14) holds if and only if $\eta(x)<\tau/2$. The latter is true due to Lemma 7.

We now show that the denominator of the LHS of Eq. (14) is indeed positive. We write this as the following:

$$1-\eta^*(x)(\tau-\eta^*(x))(1+x)=(\eta^*(x))^2(1+x)-\eta^*(x)\tau(1+x)+1 \tag{15}$$

If we take x as fixed, the above is a quadratic with respect to $\eta^*(x)$. The roots of this quadratic are $\frac{1}{2}(\tau\pm\frac{1}{2}\sqrt{\tau^2-4/(1+x)})$. If the roots exist, note that the smaller root is larger than $\frac{1}{4}\tau$; if $\eta^*(x)$ is smaller than the root, then (15) is positive. From Lemma 7, $\eta^*(x)<0.15\tau$, hence this is always true. If the roots do not exist, then (15) is always positive for any $\eta^*(x)$. Therefore, (15) is positive and hence $B'(x)>0$.

We next prove that $xB'(x)$ (and therefore, $bxB'(x)$) is increasing. Notice that the optimal effort is given by

$$\eta^*(x)=\frac{1}{a}\Phi'\left(\left(\tau-\eta^*(x)\right)\sqrt{1+x}\right)\sqrt{1+x}. \tag{16}$$

Taking the first-order derivative of the above expression results in

$$\eta^*_x(x)=\frac{\frac{1}{2}\eta^*(x)\left(-\left(\tau-\eta^*(x)\right)^2+\frac{1}{1+x}\right)}{1-\eta^*(x)(\tau-\eta^*(x))(1+x)}. \tag{17}$$

Now taking the first-order derivative of $xB'(x)$ with respect to x yields

$$\begin{aligned}\frac{d}{dx}\left(xB'(x)\right)=&\frac{\tau-\eta^*(x)}{2\sqrt{1+x}}\\&-\frac{\sqrt{1+x}\left(\frac{1}{1+x}-(\tau-\eta^*(x))^2\right)\eta^*(x)}{2\left(1-(1+x)(\tau-\eta^*(x))\eta^*(x)\right)}\\&+x\left(-\frac{\tau-\eta^*(x)}{4(1+x)^{\frac{3}{2}}}-\frac{\frac{1}{1+x}-(\tau-\eta^*(x))^2\eta^*(x)}{4\sqrt{1+x}\left(1-(1+x)(\tau-\eta^*(x))\eta^*(x)\right)}-\frac{\eta^*_x(x)}{2\sqrt{1+x}}\right.\\&-\frac{\sqrt{1+x}\left(\frac{1}{1+x}-(\tau-\eta^*(x))^2\right)\eta^*_x(x)}{2\left(1-(1+x)(\tau-\eta^*(x))\eta^*(x)\right)}\\&-\frac{\sqrt{1+x}\eta^*(x)\left(-\frac{1}{(1+x)^2}+2(\tau-\eta^*(x))\eta^*_x(x)\right)}{2\left(1-(1+x)(\tau-\eta^*(x))\eta^*(x)\right)}\\&\left.+\frac{\sqrt{1+x}\left(\frac{1}{1+x}-(\tau-\eta^*(x))^2\right)\eta^*(x)\left(-(\tau-\eta^*(x))\eta^*(x)-(1+x)(\tau-\eta^*(x))\eta^*_x(x)+(1+x)\eta^*(x)\eta^*_x(x)\right)}{2\left(1-(1+x)(\tau-\eta^*(x))\eta^*(x)\right)^2}\right).\end{aligned}$$

Using (17) in the above expression results in an equation that only involves $\eta^*(x)$ and not its derivative:

$$\begin{aligned}&\frac{-\eta^*(x)+\tau}{2\sqrt{1+x}}-\frac{\eta^*(x)\sqrt{1+x}\left(-(-\eta^*(x)+\tau)^2+\frac{1}{1+x}\right)}{2\left(1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)\right)}\\&+x\left(-\frac{-\eta^*(x)+\tau}{4(1+x)^{\frac{3}{2}}}-\frac{\eta^*(x)\sqrt{1+x}\left(-(-\eta^*(x)+\tau)^2+\frac{1}{1+x}\right)^2}{4\left(1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)\right)^2}-\frac{\eta^*(x)\left(-(-\eta^*(x)+\tau)^2+\frac{1}{1+x}\right)}{2\sqrt{1+x}\left(1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)\right)}\right.\end{aligned}$$

$$-\frac{\eta^*(x)\sqrt{1+x}\left(-\frac{1}{(1+x)^2}+\frac{\eta^*(x)(-\eta^*(x)+\tau)\left(-(-\eta^*(x)+\tau)^2+\frac{1}{1+x}\right)}{1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)}\right)}{2\left(1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)\right)}$$
$$+\eta^*(x)\sqrt{1+x}\left(-(-\eta^*(x)+\tau)^2+\frac{1}{1+x}\right)$$
$$\frac{-\eta^*(x)(-\eta^*(x)+\tau)+\frac{\eta^*(x)^2(1+x)\left(-(-\eta^*(x)+\tau)^2+\frac{1}{1+x}\right)}{2\left(1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)\right)}-\frac{\eta^*(x)(-\eta^*(x)+\tau)(1+x)\left(-(-\eta^*(x)+\tau)^2+\frac{1}{1+x}\right)}{2\left(1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)\right)}}{2\left(1-\eta^*(x)(-\eta^*(x)+\tau)(1+x)\right)^2}\Bigg). \tag{18}$$

Now, as $a\to\infty$, we argue that $\eta^*(x)\to 0$ uniformly. To see this, we make use of Lemma 7. In particular, we can write

$$\begin{aligned}\eta^*(x) =& \frac{1}{a}\Phi'\left(\left(\tau-\eta^*(x)\right)\sqrt{1+x}\right)\sqrt{1+x}\\ \overset{(a)}{\le}& \frac{1}{a}\Phi'\left(.85\tau\sqrt{1+x}\right)\sqrt{1+x}\end{aligned}$$

where (a) follows from Lemma 7. Noting that $\Phi'\left(.85\tau\sqrt{1+x}\right)\sqrt{1+x}$ is uniformly bounded and thus as $a\to\infty$, we have that $\eta^*(x)\to 0$ uniformly. Therefore, in the limit as $a\to\infty$, (18) becomes

$$\frac{\tau(2+x)}{4(1+x)^{3/2}},$$

which is positive. Therefore, for large enough a, Claim 1 holds. ■

Claim 2: There exists $\tilde{\sigma}^2$ such that the optimal effort $\eta^*(\sigma_t^2)$ for $\sigma_t^2\ge\tilde{\sigma}^2$ is decreasing.

Proof of Claim 2: Recall that the first-order condition gives us

$$\eta^*(x)=\frac{1}{a}\Phi'((\tau-\eta^*(x))\sqrt{1+x})\sqrt{1+x}.$$

Notice that $\eta^*(x)$ appears on both sides of the above equation, and we are interested in evaluating its derivative $\eta_x^*(x)$. In this regard, we take the derivative of both sides of the above equation yields

$$\begin{aligned}a\eta_x^*(x) =& \Phi''\left((\tau-\eta^*(x))\sqrt{1+x}\right)\left(-\eta_x^*(x)\sqrt{1+x}+(\tau-\eta^*(x))\frac{1}{2\sqrt{1+x}}\right)\sqrt{1+x}\\ &+\Phi'((\tau-\eta^*(x))\sqrt{1+x})\frac{1}{2\sqrt{1+x}}.\end{aligned}$$

Simplifying the above expression and using $\Phi''(y)=-y\Phi'(y)$ results in

$$\begin{aligned}\eta_x^*(x) =& \frac{1}{2}\frac{-\left(\tau-\eta^*(x)\right)^2\sqrt{1+x}+(1+x)^{-1/2}}{\frac{a}{\Phi'((\tau-\eta^*(x))\sqrt{1+x})}-(\tau-\eta^*(x))(1+x)^{3/2}}\\ =& \frac{\eta^*(x)\left(\frac{-1}{2}\left(\tau-\eta^*(x)\right)^2+\frac{1}{2(1+x)}\right)}{1-\eta^*(x)(\tau-\eta^*(x))(1+x)}\\ \overset{(a)}{\le}& \eta^*(x)\frac{-\frac{1}{8}\tau^2+\frac{1}{16}\tau^2}{\frac{1}{2}}<0,\end{aligned}$$

where (a) follows by taking x large enough so that $\frac{1}{1+x}\le\frac{\tau^2}{8}$ and $\eta^*(x)\le\min\{\frac{1}{2\tau(1+x)},\frac{\tau}{2}\}$. ■

We now proceed with the proof of the theorem. Similar to the proof of Theorem 1, we need to show that for large enough x, we have $f'(x)>1$ where

$$f(x)=\frac{1}{P^b\left(1-\Phi((\tau-\eta^*(x))\sqrt{1+x})\right)^b}$$

and

$$\eta^*(x) = \arg\max_{\eta \geq 0} 1 - \Phi((\tau - \eta)\sqrt{1+x}) - \frac{a}{2}\eta^2.$$

The derivative of $f(x)$ can be written as

$$\begin{aligned}
f'(x) &= \frac{b\Phi'((\tau-\eta^*(x))\sqrt{1+x})\left(-\eta^*_x(x)\sqrt{1+x} + (\tau-\eta^*(x))\frac{1}{2\sqrt{1+x}}\right)}{P^b\left(1-\Phi((\tau-\eta^*(x))\sqrt{1+x})\right)^{b+1}} \\
&\overset{(a)}{\geq} \frac{b\Phi'((\tau-\eta^*(x))\sqrt{1+x})\left((\tau-\eta^*(x))\frac{1}{2\sqrt{1+x}}\right)}{P^b\left(1-\Phi((\tau-\eta^*(x))\sqrt{1+x})\right)^{b+1}} \\
&\overset{(b)}{\geq} \frac{b(\tau-\eta^*(x)\sqrt{1+x})^{b+2}}{P^b 2(1+x)\Phi'((\tau-\eta^*(x))\sqrt{1+x})^b} \\
&\overset{(c)}{=} \frac{b((\tau-\eta^*(x))\sqrt{1+x})^{b+2}(1+x)^b}{2P^b a^b \eta^*(x)^b} \overset{(d)}{>} 1
\end{aligned}$$

where (a) follows from invoking Claim 2 for large enough x, (b) follows from $1-\Phi(t) < \frac{\Phi'(t)}{t}$, (c) follows from (16), and (d) follows from the fact that for large enough x, $\eta^*(x)$ goes to zero. ■

Proof of Proposition 6

Fix an instance where $\sigma_*^2 < \infty$. From Lemma 6 the first-order condition for the optimal effort is

$$\eta^*(\sigma^2) = \frac{1}{a}\Phi'((\tau-\eta^*(\sigma^2))\sqrt{1+\sigma^2})\sqrt{1+\sigma^2}. \tag{19}$$

For any fixed σ^2, if $\eta = 0$, then the RHS is positive, and hence this does not satisfy the above equation. Therefore, $\eta^*(\sigma_*^2) > 0$.

On the other hand, since $\eta^*(\sigma^2) < 0.85\tau$ by Lemma 7, the RHS of (19) is upper bounded by $\frac{1}{a}\Phi'(0.15\tau\sqrt{1+\sigma^2})\sqrt{1+\sigma^2}$, which approaches 0 as $\sigma^2 \to \infty$. Therefore, $\lim_{\sigma^2\to\infty}\eta^*(\sigma^2) = 0$. ■

Proof of Proposition 7

The sequence of noise variances are $\{\bar{\sigma}_t^2\}_{t=1}^\infty$ such that

$$\bar{\sigma}_{t+1}^2 = \alpha f(\bar{\sigma}_t^2) + (1-\alpha)\bar{\sigma}_t^2 \quad \text{for all } t,$$

where $f(\sigma^2) = \frac{1}{P^b \cdot q(\tau;\sigma^2)^b}$, same as the baseline model. Therefore, the convergence properties of the sequence $\{\bar{\sigma}_t^2\}_{t=1}^\infty$ are the same as the original sequence. ■

Proof of Proposition 8

First, suppose $M < \bar{\sigma}^2$. Then $f(x;P,\tau)$ intersects the line $y = x$ once at $x = \min\{M, \sigma_*^2\}$, as seen in the left plot of Fig. 6. Therefore, there is one active steady state at $\min\{M, \sigma_*^2\}$. Next, assume $M \geq \bar{\sigma}^2$. Then, the last intersection of $f(x;P,\tau)$ with $y = x$ occurs at $x = M$ (as seen in the right plot of Fig. 6), and this is a stable steady state. Hence in this case, there are two active stable steady states at σ_*^2 and at M. The proof of the dynamics follows from the same reasoning as Theorem 1. ■

Appendix D: Alternative Update Rule: Feature Informativeness

We present a microfoundation that motivates the noise update rule (2) in Section 6.4.

Suppose $O_t \in \mathbb{R}^k$ represents $k \in \mathbb{N}$ different features about the applicant. However, only one of these features is informative about the applicant's skill, represented by $Z \in [k]$. However, the firm does not know Z. The observation O_t takes the form:

$$O_t = \begin{cases} (S_t + \delta, \varepsilon_1, \ldots, \varepsilon_{k-1}) & \text{if } Z = 1, \\ (\varepsilon_1, S_t + \delta, \ldots, \varepsilon_{k-1}) & \text{if } Z = 2, \\ \quad\vdots & \\ (\varepsilon_1, \varepsilon_2, \ldots, S_t + \delta) & \text{if } Z = k, \end{cases}$$

where $\delta \sim \mathcal{N}(0, \sigma^2)$ and $\varepsilon_1, \ldots, \varepsilon_{k-1} \sim \mathcal{N}(0, 1 + \sigma^2)$ independently, for some parameter $\sigma^2 > 0$. We still assume that $S_t \sim \mathcal{N}(0,1)$. Note that each element of O_t has the same distribution, which is $\mathcal{N}(0, 1 + \sigma^2)$.

Belief and dynamics. Let $\rho_t(j) \in [0,1]$ be the firm's belief at time t on the probability that j is the informative feature, $Z = j$. We assume that Z is drawn uniformly at random from $[k]$, and hence $\rho_1(j) = 1/k$ for all $j \in [k]$. At time t, using the current belief $(\rho_t(j))_{j \in [k]}$, the firm computes the inferred skill $\mathbb{E}[S_t \mid O_t]$ for each applicant and hires them if and only if $\mathbb{E}[S_t \mid O_t] > \tau$. Once an applicant is hired, the firm gets to observe their true skill, S_t. Then, the firm uses the observations (O_t, S_t) for all hired applicants and uses this to update their belief on which feature is informative for the next round, $(\rho_{t+1}(j))_{j \in [k]}$.

We show that the update rule given by (2) approximates the dynamics resulting from the above model.

Proposition 10. *At round t, there exists constants $c_1, c_2 > 0$, $b(\rho_t) > 0$ which depends on $\rho_t(j)$, and $a \in (0,1)$ such that for any $\epsilon_1, \epsilon_2 > 0$ and $\delta \in (0,1)$, there exists an n_0 such that if $n \geq n_0$ applicants are hired at time t, then $\Phi_{t+1} = \mathcal{N}(0, \sigma^2)$, where with probability at least $1 - \delta$,*

$$\sigma^2 \in \left(c_1 + \frac{c_2}{1 + b(\rho_t)\,(a + \epsilon_1)^n} - \epsilon_2, c_1 + \frac{c_2}{1 + b(\rho_t)\,(a - \epsilon_1)^n} + \epsilon_2 \right).$$

Relation to (2)**.** The dynamics of the above feature informativeness model is approximated by the update rule in (2), where we interpret Pq_{t-1} as the number of applicants hired, n. By doing so, we are simplifying two aspects. First, we are considering a deterministic version of the underlying stochastic system (i.e., we assume $\epsilon_1 = \epsilon_2 = 0$ in Proposition 10). Second, we replace $c(\rho_t)$ with a constant c_3, which implies that the learning of Z does not accumulate over time. This is analogous to the assumption in Section 2 that the noise is only a function of the hires in the previous time step; we assume the firm learns how to evaluate applicants at time t from the those hired from time $t-1$, but the firm does not leverage its learnings from time $t-2$ or further back.

We next provide the proof of Proposition 10 and that of Theorem 7.

Proof of Proposition 10

Fix a round t. Let n be the number of hired applicants at round t, and let $(O_i, S_i)_{i \in [n]}$ be the observations and skills of these hired applicants. Given the prior belief $(\rho_t(j))_{j \in [k]}$, the posterior belief for Z can be written as:

$$P(Z = j \mid (O_i, S_i)_{i=1}^n) = \frac{\rho_t(j) P\left((O_i, S_i)_{i=1}^n \mid Z = j\right)}{\sum_{j'=1}^m \rho_t(j') P\left((O_i, S_i)_{i=1}^n \mid Z = j'\right)}$$

$$= \frac{1}{1+\sum_{j'=1,j'\neq j}^{n} \frac{\rho_t(j')}{\rho_t(j)} \frac{P\left((O_i,S_i)_{i=1}^n | Z=j'\right)}{P\left((O_i,S_i)_{i=1}^n | Z=j\right)}}$$

$$= \frac{1}{1+\sum_{j'=1,j'\neq j}^{n} \frac{\rho_t(j')}{\rho_t(j)} \exp\left(-\log \Lambda_n(j,j')\right)},$$

where $\Lambda_n(j,j') = \frac{P\left((O_i,S_i)_{i=1}^n | Z=j\right)}{P\left((O_i,S_i)_{i=1}^n | Z=j'\right)}$ is the likelihood ratio of the observations under $Z=j$ and $Z=j'$. Using the independence of signals across different applicants, we can write $\log\Lambda_n(j,j')$ as

$$\log\Lambda_n(j,j') = \sum_{i=1}^{n} \log \frac{P\left(O_i,S_i \mid Z=j\right)}{P\left(O_i,S_i \mid Z=j'\right)}.$$

Let $j^* \in [k]$ be the true value of Z. We can use the law of large numbers to write

$$\lim_{n\to\infty} \frac{1}{n}\Lambda_n(j,j') \xrightarrow{P} \mathbb{E}_{(O,S)\sim P(\cdot,\cdot|Z=j^*)}\left[\log\left(\frac{P\left(O,S \mid Z=j\right)}{P\left(O,S \mid Z=j'\right)}\right)\right].$$

Therefore, for any $\epsilon,\delta>0$, there exists an n_0 such that if $n\geq n_0$, $\frac{1}{n}\Lambda_n(j,j')$ is within ϵ of its expectation with probability at least $1-\delta$.

We now evaluate the expectation on the right hand side, considering three cases. Denote by P_j the distribution of (O,S) when $Z=j$. First, suppose $j=j^*$ and $j'\neq j^*$. Then, we have

$$\mathbb{E}_{(O,S)\sim P(\cdot,\cdot|Z=j^*)}\left[\log\left(\frac{P\left(O,S \mid Z=j\right)}{P\left(O,S \mid Z=j'\right)}\right)\right] = D_{\mathrm{KL}}\left(P_{j^*}||P_{j'}\right) > 0,$$

where D_{KL} is the Kullback–Leibler divergence. Similarly, if $j'=j^*$ and $j\neq j'$, then

$$\mathbb{E}_{(O,S)\sim P(\cdot,\cdot|Z=j^*)}\left[\log\left(\frac{P\left(O,S \mid Z=j\right)}{P\left(O,S \mid Z=j'\right)}\right)\right] = -D_{\mathrm{KL}}\left(P_{j^*}||P_{j'}\right) < 0,$$

Lastly, if $j,j'\neq j^*$, then

$$\mathbb{E}_{(O,S)\sim P(\cdot,\cdot|Z=j^*)}\left[\log\left(\frac{P\left(O,S \mid Z=j\right)}{P\left(O,S \mid Z=j'\right)}\right)\right] = 0.$$

This is because if $Z=j^*$, and $j,j'\neq j^*$, then the j and j'th entries of O is pure noise, and these entries are independent and have the same distribution. Therefore, for any realization of (O,S) where the ratio $\frac{P(O,S|Z=j)}{P(O,S|Z=j')}$ corresponds to a/b, there is a realization (O',S') with the same likelihood where the ratio $\frac{P\left(O',S'|Z=j\right)}{P(O',S'|Z=j')} = b/a$. These two samples cancel out to 0, and this occurs for all samples.

Let $D = D_{\mathrm{KL}}\left(P_{j^*}||P_{j'}\right) > 0$ for $j'\neq j^*$. Fix $\epsilon\in(0,D)$ and $\delta>0$. There exists an n_0 such that if $n>0$, the following two expressions hold with probability greater than $1-\delta$. First, if $j=j^*$, then

$$\rho_{t+1}(j) \in \left(\frac{1}{1+C_j(\rho_t)(e^{-D+\epsilon})^n}, \frac{1}{1+C_j(\rho_t)(e^{-D-\epsilon})^n}\right), \tag{20}$$

where $C_j(\rho_t) = \sum_{j'=1,j'\neq j}^{n} \frac{\rho_t(j')}{\rho_t(j)}$. Second, if $j\neq j^*$, then

$$\rho_{t+1}(j) \leq \frac{1}{1+\frac{\rho_t(j)}{\rho_t(j')}(e^{D-\epsilon})^n}. \tag{21}$$

We now consider the distribution of $\mathbb{E}[S_{t+1} \mid O_{t+1}]$. When the firm observes O_{t+1}, the inferred skill can be written as

$$\mathbb{E}[S_{t+1} \mid O_{t+1}] = \sum_{j=1}^{k} \mathbb{E}[S_{t+1} \mid O_{t+1}, Z_{t+1} = j] \Pr(Z_{t+1} = j \mid O_{t+1}).$$

Since the distribution of O_{t+1} is equal under any value of Z_{t+1}, we have $\Pr(Z_{t+1} = j \mid O_{t+1}) = \Pr(Z_{t+1} = j) = \rho_{t+1}(j)$. If $O_{t+1}(j)$ is the j'th element of O_{t+1}, then

$$\mathbb{E}[S_{t+1} \mid O_{t+1}] = \sum_{j=1}^{k} \rho_{t+1}(j) \frac{O_{t+1}(j)}{1+\sigma^2}.$$

Since $O_{t+1}(j) \sim \mathcal{N}(0, 1+\sigma^2)$, the distribution of $\mathbb{E}[S_{t+1} \mid O_{t+1}]$ is then

$$\mathbb{E}[S_{t+1} \mid O_{t+1}] \sim \mathcal{N}\left(0, \frac{1}{1+\sigma^2} \sum_{j=1}^{k} \rho_{t+1}(j)^2\right).$$

Therefore, we need to show that $\gamma = \frac{1}{1+\sigma^2} \sum_{j=1}^{k} \rho_{t+1}(j)^2$ has the form stated in the result.

Consider the sum $\sum_{j=1}^{k} \rho_{t+1}(j)^2$. For $j \neq j^*$, we have

$$\rho_{t+1}(j)^2 \leq \frac{1}{(1 + \frac{\rho_t(j)}{\rho_t(j')} (e^{D-\epsilon})^n)^2}.$$

Note that $e^{D-\epsilon} > 1$, and hence the term $(e^{D-\epsilon})^n$ grows exponentially large in n. Therefore, $\rho_{t+1}(j)^2 \to 0$ as $n \to \infty$, and hence this term can be incorporated into ϵ_2. For $j = j^*$, we have

$$\rho_{t+1}(j^*)^2 \leq \frac{1}{1 + 2C_j(\rho_t)(e^{-D-\epsilon})^n + C_j(\rho_t)^2 (e^{-D-\epsilon})^{2n}}$$

In the denominator, the second term represents the expression $c(\rho_t)(a-\epsilon_1)^n$, where $c(\rho_t) = 2C_j(\rho_t)$, and $a = e^{-D}$. The third term is negligible compared to the second term for large n, and hence can be incorporated into ϵ_1. This shows the upper bound $\sigma^2 \leq c_1 + \frac{c_2}{1+c(\rho_t)\,(a-\epsilon_1)^n} + \epsilon_2$. We can use the same analogous arguments to show the lower bound, and this proves the desired result.

Proof of Theorem 7

Recall that the distribution of $\mathbb{E}[S_t \mid O_t]$ takes the form :

$$\mathbb{E}[S_t \mid O_t] \sim N\left(0, c_1 + \frac{c_2}{1 + c_3\, a^{P q_{t-1}}}\right).$$

First, we assume $c_1 = 0$. Let $\gamma_t = b\, a^{P q_{t-1}}$. We will characterize how the term γ_t changes over time. At time t, the proportion of the population hired is

$$q_t = \Pr\left(N\left(0, \frac{c_2}{1+\gamma_t}\right) > \tau\right) = 1 - \Phi\left(\tilde{\tau}\sqrt{1+\gamma_t}\right),$$

where $\tilde{\tau} = \frac{\tau}{\sqrt{c_2}}$. Let $w(x)$ be the function that maps the γ_t value from one time step to the next, which takes the following form:

$$w(x) = b \cdot a^{P \cdot \left(1 - \Phi\left(\tau\sqrt{1+x}\right)\right)}. \tag{22}$$

Our goal is to understand how $w(x)$ behaves in relation to $y = x$. If $w(x) > x$, then the γ_t value increases, while if $w(x) < x$, then the γ_t values decreases. $w(x) = x$ is a steady state.

We take the derivative of $w(x)$:

$$w'(x) = b \cdot \log a \cdot a^{P\cdot\left(1-\Phi\left(\tilde{\tau}\sqrt{1+x}\right)\right)} \cdot P \cdot (-\phi(\tilde{\tau}\sqrt{1+x})) \cdot \tilde{\tau} \cdot \frac{1}{2}(1+x)^{-1/2} \tag{23}$$

$$= \frac{b\tilde{\tau}P\log(1/a)}{2} \cdot \frac{a^{P\cdot\left(1-\Phi\left(\tilde{\tau}\sqrt{1+x}\right)\right)} \cdot \phi(\tilde{\tau}\sqrt{1+x})}{\sqrt{1+x}}. \tag{24}$$

It is easy to see that $w'(x)$ is non-negative for all $x \geq 0$. We define the space of parameters $\Theta \in (0,1) \times \mathbb{R}_+$:

$$\Theta = \left\{(a,b) \,:\, b \geq B,\, \exp(-b) \leq a \leq b^{-C}\right\}$$

for constants $B > 0, C \geq \frac{2}{1-\Phi(\tau)}$. We show that an instance with these parameters have two distinct steady states.

Proposition 11. *If $(a,b) \in \Theta$, there exist two points $x_1 < x_2$ where $w(x_i) = x_i$, and there exists a $\delta > 0$ such that $w(x) > x_i$ for $x \in (x_i - \delta, x_i)$ and $w(x) < x_i$ for $x \in (x_i, x_i + \delta)$ for both $i = 1, 2$.*

Proposition 11 says that there are at least two intersections of $w(x)$ with $y = x$ where $w(x)$ approaches from above ($w(x) > x$ right before the intersection, but $w(x) < x$ right after the intersection). These intersections denote a stable steady state, and this result says that there are at least two distinct steady states.

We will show that there is a parameter regime for a and b where the following conditions hold:

(a) $w(0) > 0$.

(b) $w(0) \leq 0.01$.

(c) $w'(x) \leq 1/2$ for all $x \in [0, 0.1]$.

(d) $w(0.98b) \geq 0.99b$.

We now show that properties (a)-(d) hold when $(a,b) \in \Theta$. Property (a) holds for any $a, b > 0$.

Property (d). Let $B_1 > 0$ such that for all $b \geq B_1$,

$$a \geq 0.99^{\frac{1}{1-\Phi(\tilde{\tau}\sqrt{1+0.98b})}}. \tag{25}$$

Such a B_1 exists since $a \geq \exp(-b)$, and the RHS of (25) tends to 0 super-exponentially. Then,

$$\begin{aligned} w(0.98b) &= b \cdot a^{1-\Phi(\tilde{\tau}\sqrt{1+0.98b})} \\ &\geq b \cdot 0.99^{\frac{1-\Phi(\tilde{\tau}\sqrt{1+0.98b})}{1-\Phi(\tilde{\tau}\sqrt{1+0.98b})}} \\ &= 0.99b. \end{aligned}$$

Property (b). Since $C \geq 2 \cdot \frac{1}{1-\Phi(\tilde{\tau})}$, there exists $B_2 > 0$ such that if $b \geq B_2$,

$$a \leq \left(\frac{1}{b}\right)^C \leq \left(\frac{0.01}{b}\right)^{\frac{1}{1-\Phi(\tilde{\tau})}}.$$

Then, property (b) follows:

$$\begin{aligned} w(0) &= b \cdot a^{1-\Phi(\tilde{\tau})} \\ &\leq b \cdot \left(\frac{0.01}{b}\right)^{\frac{1-\Phi(\tilde{\tau})}{1-\Phi(\tilde{\tau})}} \\ &\leq 0.01. \end{aligned}$$

Let $B = \max\{B_1, B_2\}$.

Property (c). Since $a \geq \exp(-b)$, then $\log(1/a) \leq b$. Using this and the upper bound on a, we have

$$\begin{aligned} w'(x) &= \frac{b\tilde{\tau}\log(1/a)}{2} \cdot \frac{a^{\left(1-\Phi\left(\tilde{\tau}\sqrt{1+x}\right)\right)} \cdot \phi(\tilde{\tau}\sqrt{1+x})}{\sqrt{1+x}} \\ &\leq \frac{b^2\tilde{\tau}}{2} \cdot \frac{b^{-C\left(1-\Phi\left(\tilde{\tau}\sqrt{1+x}\right)\right)} \cdot \phi(\tilde{\tau}\sqrt{1+x})}{\sqrt{1+x}} \\ &= \frac{\tilde{\tau}\phi(\tilde{\tau}\sqrt{1+x})}{2\sqrt{1+x}} \cdot b^{2-C\left(1-\Phi\left(\tilde{\tau}\sqrt{1+x}\right)\right)} \end{aligned} \tag{26}$$

For all $x \in [0, 0.1]$, the first term in (26) is bounded below and above by a constant, and the expression $(1-\Phi\left(\tilde{\tau}\sqrt{1+x}\right))$ is also bounded below and above. Then, one can choose C large enough so that the entire expression in (26) is less than $1/2$ for all $b \geq B$.

Proof of Proposition 11. We provide the proof of Proposition 11 given that the above conditions hold. (a) states that the function starts positive, so $w(x)$ starts above $y = x$. Combining (b) and (c) implies that $w(0.1) \leq 0.001 + \frac{1}{2}0.1 < 0.1$. Therefore, there is an intersection of $w(x)$ and $y = x$ at some point $x_1 \in (0, 0.1)$.

Next, we will show that the second intersection x_2 is close to b. Note that $w(x) \leq b$ for all $x \geq 0$. Since $w(0.98b) \geq 0.99b > 0.98b$, $w(x)$ is larger than $y = x$ when $x = 0.98b$. But when $x \geq b$, we have $w(x) \leq b$. So there must be an intersection of $w(x)$ with $y = x$ for $x_2 \in (0.98b, b)$. □

Lastly, Proposition 11 assumes that $c_1 = 0$. Since the impact of c_1 on q_t is continuous, it also impacts the function $w(s)$ in a continuous fashion. Therefore, if $c_1 > 0$ is small enough, the steady states will remain the same.